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MANSFIELD MERRIMAN AND ROBERT S. WOODWARD.

No. 8.

VECTOR ANALYSIS

AND

QUATERNIONS.

BY

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EDITORS' PREFACE.

THE volume called Higher Mathematics, the first edition of which was published in 1896, contained eleven chapters by eleven authors, each chapter being independent of the others, but all supposing the reader to have at least a mathematical training equivalent to that given in classical and engineering colleges. The publication of that volume is now discontinued and the chapters are issued in separate form. In these reissues it will generally be found that the monographs are enlarged by additional articles or appendices which either amplify the former presentation or record recent advances. This plan of publication has been arranged in order to meet the demand of teachers and the convenience of classes, but it is also thought that it may prove advantageous to readers in special lines of mathematical literature.

It is the intention of the publishers and editors to add other monographs to the series from time to time, if the call for the same seems to warrant it. Among the topics which are under consideration are those of elliptic functions, the theory of numbers, the group theory, the calculus of variations, and non-Euclidean geometry; possibly also monographs on branches of astronomy, mechanics, and mathematical physics may be included. It is the hope of the editors that this form of publication may tend to promote mathematical study and research over a wider field than that which the former volume has occupied.

December, 1905.

AUTHOR'S PREFACE.

SINCE this Introduction to Vector Analysis and Quaternions was first published in 1896, the study of the subject has become much more general; and whereas some reviewers then regarded the analysis as a luxury, it is now recognized as a necessity for the exact student of physics or engineering. In America, Professor Hathaway has published a Primer of Quaternions (New York, 1896), and Dr. Wilson has amplified and extended Professor Gibbs' lectures on vector analysis into a text-book for the use of students of mathematics and physics (New York, 1901). In Great Britain, Professor Henrici and Mr. Turner have published a manual for students entitled Vectors and Rotors (London, 1903); Dr. Knott has prepared a new edition of Kelland and Tait's Introduction to Quaternions (London, 1904); and Professor Joly has realized Hamilton's idea of a Manual of Quaternions (London, 1905). In Germany Dr. Bucherer has published Elemente der Vektoranalysis (Leipzig, 1903) which has now reached a second edition.

Also the writings of the great masters have been rendered more accessible. A new edition of Hamilton's classic, the Elements of Quaternions, has been prepared by Professor Joly (London, 1899, 1901); Tait's Scientific Papers have been reprinted in collected form (Cambridge, 1898, 1900); and a complete edition of Grassmann's mathematical and physical works has been edited by Friedrich Engel with the assistance of several of the eminent mathematicians of Germany (Leipzig, 1894–). In the same interval many papers, pamphlets, and discussions have appeared. For those who desire information on the literature of the subject a Bibliography has been published by the Association for the promotion of the study of Quaternions and Allied Mathematics (Dublin, 1904).

There is still much variety in the matter of notation, and the relation of Vector Analysis to Quaternions is still the subject of discussion (see Journal of the Deutsche Mathematiker-Vereinigung for 1904 and 1905).

CHATHAM, ONTARIO, CANADA, December, 1905.

CONTENTS.

VECTOR ANALYSIS AND QUATERNIONS.

ART. 1. INTRODUCTION.

By "Vector Analysis" is meant a space analysis in which the vector is the fundamental idea; by "Quaternions" is meant a space-analysis in which the quaternion is the fundamental idea. They are in truth complementary parts of one whole; and in this chapter they will be treated as such, and developed so as to harmonize with one another and with the Cartesian Analysis.* The subject to be treated is the analysis of quantities in space, whether they are vector in nature, or quaternion in nature, or of a still different nature, or are of such a kind that they can be adequately represented by space quantities.

Every proposition about quantities in space ought to remain true when restricted to a plane; just as propositions about quantities in a plane remain true when restricted to a straight line. Hence in the following articles the ascent to the algebra of space is made through the intermediate algebra of the plane. Arts. 2–4 treat of the more restricted analysis, while Arts. 5–10 treat of the general analysis.

This space analysis is a universal Cartesian analysis, in the same manner as algebra is a universal arithmetic. By providing an explicit notation for directed quantities, it enables their general properties to be investigated independently of any particular system of coordinates, whether rectangular, cylindrical, or polar. It also has this advantage that it can express

* For a discussion of the relation of Vector Analysis to Quaternions, see Nature, 1891–1893.

the directed quantity by a linear function of the coordinates, instead of in a roundabout way by means of a quadratic function.

The different views of this extension of analysis which have been held by independent writers are briefly indicated by the titles of their works:

Argand, Essai sur une maniére de représenter les quantités imaginaires dans les constructions géométriques, 1806.

Warren, Treatise on the geometrical representation of the square roots of negative quantities, 1828.

Moebius, Der barycentrische Calcul, 1827.

Bellavitis, Calcolo delle Equipollenze, 1835.

Grassmann, Die lineale Ausdehnungslehre, 1844.

De Morgan, Trigonometry and Double Algebra, 1849.

O'Brien, Symbolic Forms derived from the conception of the translation of a directed magnitude. Philosophical Transactions, 1851.

Hamilton, Lectures on Quaternions, 1853, and Elements of Quaternions, 1866.

Tait, Elementary Treatise on Quaternions, 1867.

Hankel, Vorlesungen über die complexen Zahlen und ihre Functionen, 1867.

Schlegel, System der Raumlehre, 1872.

Hoüel, Théorie des quantités complexes, 1874.

Gibbs, Elements of Vector Analysis, 1881–4.

Peano, Calcolo geometrico, 1888.

Hyde, The Directional Calculus, 1890.

Heaviside, Vector Analysis, in "Reprint of Electrical Papers," 1885–92.

Macfarlane, Principles of the Algebra of Physics, 1891. Papers on Space Analysis, 1891–3.

An excellent synopsis is given by Hagen in the second volume of his "Synopsis der höheren Mathematik."

Art. 2. Addition of Coplanar Vectors.

By a "vector" is meant a quantity which has magnitude and direction. It is graphically represented by a line whose

length represents the magnitude on some convenient scale, and whose direction coincides with or represents the direction of the vector. Though a vector is represented by a line, its physical dimensions may be different from that of a line. Examples are a linear velocity which is of one dimension in length, a directed area which is of two dimensions in length, an axis which is of no dimensions in length.

A vector will be denoted by a capital italic letter, as B,* its magnitude by a small italic letter, as b, and its direction by a small Greek letter, as β. For example, $B = b\beta$, $R = r\rho$. Sometimes it is necessary to introduce a dot or a mark $\angle$ to separate the specification of the direction from the expression for the magnitude;† but in such simple expressions as the above, the difference is sufficiently indicated by the difference of type. A system of three mutually rectangular axes will be indicated, as usual, by the letters i, j, k.

The analysis of a vector here supposed is that into magnitude and direction. According to Hamilton and Tait and other writers on Quaternions, the vector is analyzed into tensor and unit-vector, which means that the tensor is a mere ratio destitute of dimensions, while the unit-vector is the physical magnitude. But it will be found that the analysis into magnitude and direction is much more in accord with physical ideas, and explains readily many things which are difficult to explain by the other analysis.

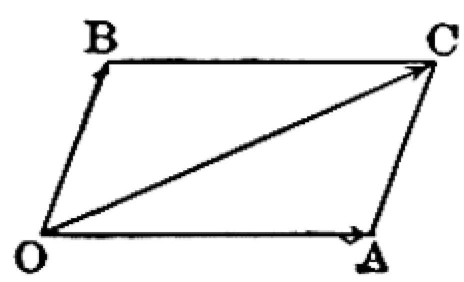
A vector quantity may be such that its components have a common point of application and are applied simultaneously; or it may be such that its components are applied in succession, each component starting from the end of its predecessor. An example of the former is found in two forces applied simultaneously at the same point, and an example of the latter in

* This notation is found convenient by electrical writers in order to harmonize with the Hospitalier system of symbols and abbreviations.

† The dot was used for this purpose in the author's Note on Plane Algebra, 1883; Kennelly has since used $\angle$ for the same purpose in his electrical papers

two rectilinear displacements made in succession to one another.

Composition of Components having a common Point of Application.—Let OA and OB represent two vectors of the same kind simultaneously applied at the point O. Draw BC parallel to OA, and AC parallel to OB, and join OC. The diagonal OC represents in magnitude and direction and point of application the resultant of OA and OB. This principle was discovered with reference to force, but it applies to any vector quantity coming under the above conditions.



Take the direction of OA for the initial direction; the direction of any other vector will be sufficiently denoted by the angle round which the initial direction has to be turned in order to coincide with it. Thus OA may be denoted by $f_1\underline{/0}$, OB by $f_2\underline{/\theta_2}$, OC by $f\underline{/\theta}$. From the geometry of the figure it follows that

$$f^2 = f_1^2 + f_2^2 + 2f_1f_2\cos\theta_2$$

and
$$\tan\theta = \frac{f_2\sin\theta_2}{f_1 + f_2\cos\theta_2};$$

hence $\mathrm{OC} = \sqrt{f_1^2 + f_2^2 + 2f_1f_2\cos\theta_2}\ \underline{/\tan^{-1}\frac{f_2\sin\theta_2}{f_1 + f_2\cos\theta_2}}$.

Example.—Let the forces applied at a point be $2\underline{/0^\circ}$ and $3\underline{/60^\circ}$. Then the resultant is $\sqrt{4+9+12\times\frac{1}{2}}\ \underline{/\tan^{-1}}\ \frac{3\sqrt{3}}{7}$ $= 4.36\underline{/36^\circ\ 30'}$.

If the first component is given as $f_1\underline{/\theta_1}$, then we have the more symmetrical formula

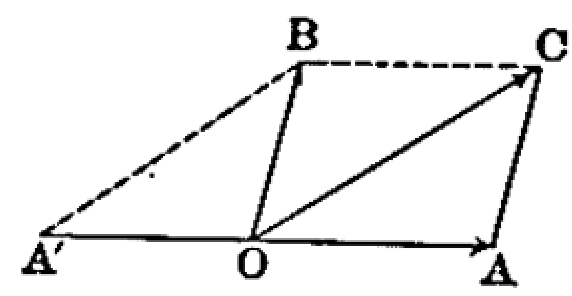
$$\mathrm{OC} = \sqrt{f_1^2 + f_2^2 + 2f_1f_2\cos(\theta_2 - \theta_1)}\ \underline{/\tan^{-1}\frac{f_1\sin\theta_1 + f_2\sin\theta_2}{f_1\cos\theta_1 + f_2\cos\theta_2}}.$$

When the components are equal, the direction of the resultant bisects the angle formed by the vectors; and the magnitude of the resultant is twice the projection of either component on the bisecting line. The above formula reduces to

$$\mathrm{OC} = 2f_1\cos\frac{\theta_2}{2}\underline{/\frac{\theta_2}{2}}.$$

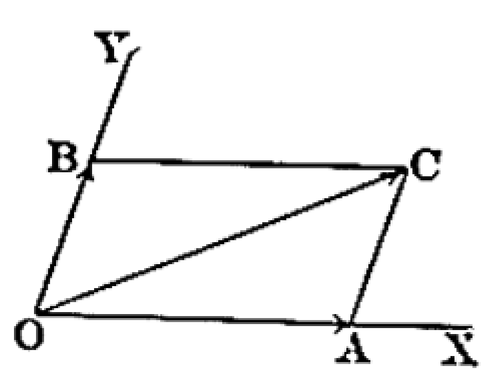
Example.—The resultant of two equal alternating electromotive forces which differ 120° in phase is equal in magnitude to either and has a phase of 60°.

Given a vector and one component, to find the other component.—Let OC represent the resultant, and OA the component. Join AC and draw OB equal and parallel to AC. The line OB represents the component required, for it is the only line which combined with OA gives OC as resultant. The line OB is identical with the diagonal of the parallelogram formed by OC and OA reversed; hence the rule is, "Reverse the direction of the component, then compound it with the given resultant to find the required component." Let $f\underline{/\theta}$ be the vector and $f_1\underline{/0}$ one component; then the other component is



$$f_2\underline{/\theta_2} = \sqrt{f^2+f_1^2 - 2ff_1 \cos\theta}\underline{/\tan^{-1}\frac{f\sin\theta}{-f_1+f\cos\theta}}.$$

Given the resultant and the directions of the two components, to find the magnitude of the components.—The resultant is represented by OC, and the directions by OX and OY. From C draw CA parallel to OY, and CB parallel to OX; the lines OA and OB cut off represent the required components. It is evident that OA and OB when compounded produce the given resultant OC, and there is only one set of two components which produces a given resultant; hence they are the only pair of components having the given directions.



Let $f\underline{/\theta}$ be the vector and $\underline{/\theta_1}$ and $\underline{/\theta_2}$ the given directions. Then

$$f_1 + f_2\cos(\theta_2 - \theta_1) = f\cos(\theta - \theta_1),$$
$$f_1\cos(\theta_2 - \theta_1) + f_2 = f\cos(\theta_2 - \theta),$$

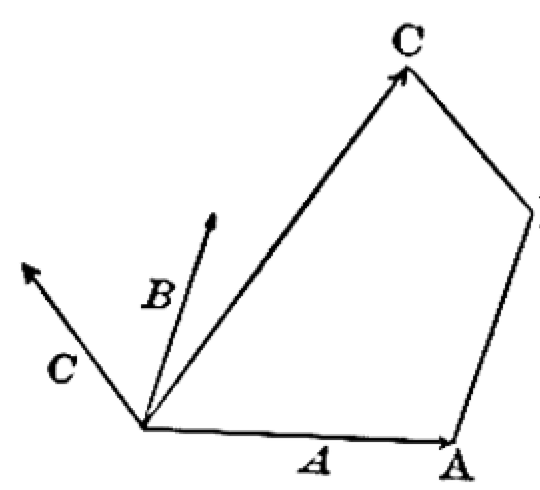
from which it follows that

$$f_1 = f\,\frac{\{\cos(\theta - \theta_1) - \cos(\theta_2 - \theta)\cos(\theta_2 - \theta_1)\}}{1 - \cos^2(\theta_2 - \theta_1)}.$$

For example, let $100\underline{/60^\circ}$, $\underline{/30^\circ}$, and $\underline{/90^\circ}$ be given; then

$$f_1 = 100 \frac{\cos 30^\circ}{1 + \cos 60^\circ}.$$

Composition of any Number of Vectors applied at a common Point.—The resultant may be found by the following graphic construction: Take the vectors in any order, as A, B, C.



From the end of A draw B' equal and parallel to B, and from the end of B' draw C' equal and parallel to C; the vector from the beginning of A to the end of C' is the resultant of the given vectors. This follows by continued application of the parallelogram construction. The resultant obtained is the same, whatever the order; and as the order is arbitrary, the area enclosed has no physical meaning.

The result may be obtained analytically as follows:

Given $$f_1\underline{/\theta_1} + f_2\underline{/\theta_2} + f_3\underline{/\theta_3} + \ldots + f_n\underline{/\theta_n}.$$

Now $$f_1\underline{/\theta_1} = f_1 \cos\theta_1\underline{/0} + f_1 \sin\theta_1 \underline{/\frac{\pi}{2}}.$$

Similarly $$f_2\underline{/\theta_2} = f_2 \cos\theta_2\underline{/0} + f_2 \sin\theta_2 \underline{/\frac{\pi}{2}},$$

and $$f_n\underline{/\theta_n} = f_n \cos\theta_n\underline{/0} + f_n \sin\theta_n \underline{/\frac{\pi}{2}}.$$

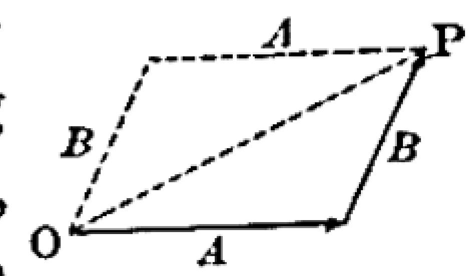
Hence $$\Sigma\{f\underline{/\theta}\} = \{\Sigma f \cos\theta\}\underline{/0} + \{\Sigma f \sin\theta\}\underline{/\frac{\pi}{2}}$$

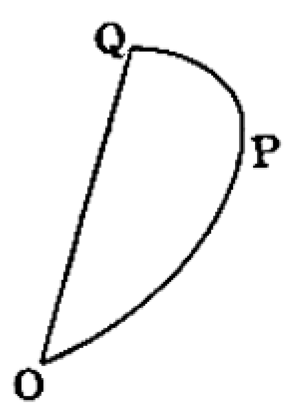
$$= \sqrt{(\Sigma f \cos\theta)^2 + (\Sigma f \sin\theta)^2}\,.\,\tan^{-1}\frac{\Sigma f \sin\theta}{\Sigma f \cos\theta}.$$

In the case of a sum of simultaneous vectors applied at a common point, the ordinary rule about the transposition of a term in an equation holds good. For example, if $A + B + C = 0$, then $A + B = -C$, and $A + C = -B$, and $B + C = -A$, etc. This is permissible because there is no real order of succession among the given components.*

* This does not hold true of a sum of vectors having a real order of succession. It is a mistake to attempt to found space-analysis upon arbitrary formal

Composition of Successive Vectors.—The composition of successive vectors partakes more of the nature of multiplication than of addition. Let A be a vector starting from the point O, and B a vector starting from the end of A. Draw the third side OP, and from O draw a vector equal to B, and from its extremity a vector equal to A. The line OP is not the complete equivalent of $A + B$; if it were so, it would also be the complete equivalent of $B + A$. But $A + B$ and $B + A$ determine different paths; and as they go oppositely around, the areas they determine with OP have different signs. The diagonal OP represents $A + B$ only so far as it is considered independent of path. For any number of successive vectors, the sum so far as it is independent of path is the vector from the initial point of the first to the final point of the last. This is also true when the successive vectors become so small as to form a continuous curve. The area between the curve OPQ and the vector OQ depends on the path, and has a physical meaning.





Prob. 1. The resultant vector is $123\underline{/45^\circ}$, and one component is $100\underline{/0^\circ}$; find the other component.

Prob. 2. The velocity of a body in a given plane is $200\underline{/75^\circ}$, and one component is $100\underline{/25^\circ}$; find the other component.

Prob. 3. Three alternating magnetomotive forces are of equal virtual value, but each pair differs in phase by 120°; find the resultant. (Ans. Zero.)

Prob. 4. Find the components of the vector $100\underline{/70^\circ}$ in the directions 20° and 100°.

Prob. 5. Calculate the resultant vector of $1\underline{/10^\circ}$, $2\underline{/20^\circ}$, $3\underline{/30^\circ}$, $4\underline{/40^\circ}$.

Prob. 6. Compound the following magnetic fluxes: $h \sin nt + h \sin (nt - 120^\circ)\underline{/120^\circ} + h \sin (nt - 240^\circ)\underline{/240^\circ}$. (Ans. $\frac{3}{2}h\underline{/nt}$.)

laws; the fundamental rules must be made to express universal properties of the thing denoted. In this chapter no attempt is made to apply formal laws to directed quantities. What is attempted is an analysis of these quantities.

Prob. 7. Compound two alternating magnetic fluxes at a point, $a \cos nt \underline{/0}$ and $a \sin nt \underline{/\frac{\pi}{2}}$. (Ans. $a \underline{/nt}$.)

Prob 8. Find the resultant of two simple alternating electromotive forces $100\underline{/20^\circ}$ and $50\underline{/75^\circ}$.

Prob. 9. Prove that a uniform circular motion is obtained by compounding two equal simple harmonic motions which have the space-phase of their angular positions equal to the supplement of the time-phase of their motions.

ART. 3. PRODUCTS OF COPLANAR VECTORS.

When all the vectors considered are confined to a common plane, each may be expressed as the sum of two rectangular components. Let i and j denote two directions in the plane at right angles to one another; then $A = a_1 i + a_2 j$, $B = b_1 i + b_2 j$, $R = xi + yj$. Here i and j are not unit-vectors, but rather signs of direction.

Product of two Vectors.—Let $A = a_1 i + a_2 j$ and $B = b_1 i + b_2 j$ be any two vectors, not necessarily of the same kind physically. We assume that their product is obtained by applying the distributive law, but we do not assume that the order of the factors is indifferent. Hence

$$AB = (a_1 i + a_2 j)(b_1 i + b_2 j) = a_1 b_1 ii + a_2 b_2 jj + a_1 b_2 ij + a_2 b_1 ji.$$

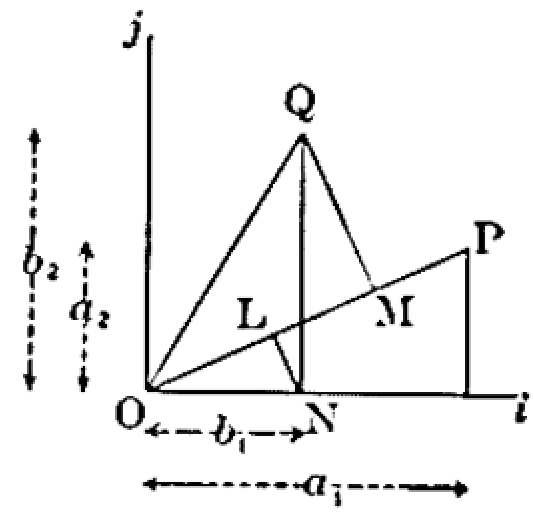
If we assume, as suggested by ordinary algebra, that the square of a sign of direction is $+$, and further that the product of two directions at right angles to one another is the direction normal to both, then the above reduces to

$$AB = a_1 b_1 + a_2 b_2 + (a_1 b_2 - a_2 b_1)k.$$

Thus the complete product breaks up into two partial products, namely, $a_1 b_1 + a_2 b_2$ which is independent of direction, and $(a_1 b_2 - a_2 b_1)k$ which has the axis of the plane for direction.*

* A common explanation which is given of $ij = k$ is that i is an operator, j an operand, and k the result. The kind of operator which i is supposed to denote is a quadrant of turning round the axis i; it is supposed not to be an axis, but a quadrant of rotation round an axis. This explains the result $ij = k$, but unfortunately it does not explain $ii = +$; for it would give $ii = i$.

Scalar Product of two Vectors.—By a scalar quantity is meant a quantity which has magnitude and may be positive or negative but is destitute of direction. The former partial product is so called because it is of such a nature. It is denoted by SAB where the symbol S, being in Roman type, denotes, not a vector, but a function of the vectors A and B. The geometrical meaning of SAB is the product of A and the orthogonal projection of B upon A. Let OP and OQ represent the vectors A and B; draw QM and NL perpendicular to OP. Then



$$
\begin{aligned}
(\mathrm{OP})(\mathrm{OM}) &= (\mathrm{OP})(\mathrm{OL}) + (\mathrm{OP})(\mathrm{LM}), \\
&= a\left\{ b_1\frac{a_1}{a} + b_2\frac{a_2}{a} \right\}, \\
&= a_1 b_1 + a_2 b_2 .
\end{aligned}
$$

Corollary 1.—SBA = SAB. For instance, let A denote a force and B the velocity of its point of application; then SAB denotes the rate of working of the force. The result is the same whether the force is projected on the velocity or the velocity on the force.

Example 1.—A force of 2 pounds East + 3 pounds North is moved with a velocity of 4 feet East per second + 5 feet North per second; find the rate at which work is done.

$$2 \times 4 + 3 \times 5 = 23 \text{ foot-pounds per second.}$$

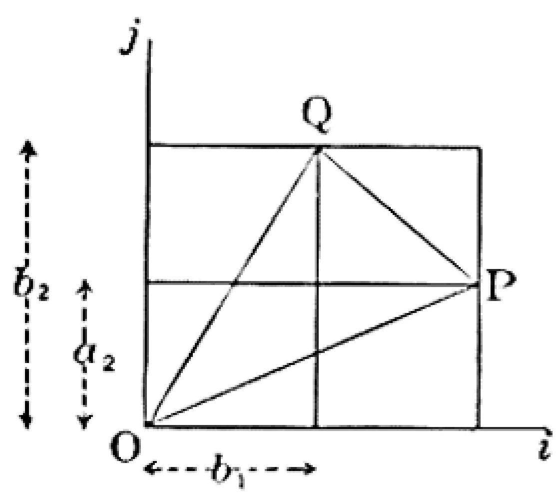
Corollary 2.—$A^2 = a_1^2 + a_2^2 = a^2$. The square of any vector is independent of direction; it is an essentially positive or signless quantity; for whatever the direction of A, the direction of the other A must be the same; hence the scalar product cannot be negative.

Example 2.—A stone of 10 pounds mass is moving with a velocity 64 feet down per second + 100 feet horizontal per second. Its kinetic energy then is

$$\frac{10}{2}(64^2 + 100^2) \text{ foot-poundals,}$$

a quantity which has no direction. The kinetic energy due to the downward velocity is $10 \times \frac{64^2}{2}$ and that due to the horizontal velocity is $\frac{10}{2} \times 100^2$; the whole kinetic energy is obtained, not by vector, but by simple addition, when the components are rectangular.

Vector Product of two Vectors.—The other partial product from its nature is called the vector product, and is denoted by VAB. Its geometrical meaning is the product of A and the projection of B which is perpendicular to A, that is, the area of the parallelogram formed upon A and B. Let OP and OQ represent the vectors A and B, and draw the lines indicated by the figure. It is then evident that the area



of the triangle OPQ $= a_1b_2 - \frac{1}{2}a_1a_2 - \frac{1}{2}b_1b_2 - \frac{1}{2}(a_1 - b_1)(b_2 - a_2)$,
$= \frac{1}{2}(a_1b_2 - a_2b_1)$.

Thus $(a_1b_2 - a_2b_1)k$ denotes the magnitude of the parallelogram formed by A and B and also the axis of the plane in which it lies.

It follows that $VBA = -VAB$. It is to be observed that the coordinates of A and B are mere component vectors, whereas A and B themselves are taken in a real order.

Example.—Let $A = (10i + 11j)$ inches and $B = (5i + 12j)$ inches, then $VAB = (120 - 55)k$ square inches; that is, 65 square inches in the plane which has the direction k for axis.

If A is expressed as $a\alpha$ and B as $b\beta$, then $SAB = ab \cos \alpha\beta$, where $\alpha\beta$ denotes the angle between the directions α and β.

Example.—The effective electromotive force of 100 volts per inch $\underline{/90^\circ}$ along a conductor 8 inch $\underline{/45^\circ}$ is $SAB = 8 \times 100$ $\cos \underline{/45^\circ}\ \underline{/90^\circ}$ volts, that is, $800 \cos 45^\circ$ volts. Here $\underline{/45^\circ}$ indicates the direction α and $\underline{/90^\circ}$ the direction β, and $\underline{/45^\circ}\ \underline{/90^\circ}$ means the angle between the direction of 45° and the direction of 90°.

Also $VAB = ab \sin \alpha\beta \,.\, \overline{\alpha\beta}$, where $\overline{\alpha\beta}$ denotes the direction which is normal to both α and β, that is, their pole.

Example.—At a distance of 10 feet $\underline{/30^\circ}$ there is a force of 100 pounds $\underline{/60^\circ}$. The moment is $\mathrm{V}AB$

$$= 10 \times 100 \sin \underline{/30^\circ}\ \underline{/60^\circ} \text{ pound-feet } \overline{90^\circ /}\ \underline{/90^\circ}$$

$$= 1000 \sin 30^\circ \text{ pound-feet } \overline{90^\circ /}\ \underline{/90^\circ}.$$

Here $\overline{90^\circ /}$ specifies the plane of the angle and $\underline{/90^\circ}$ the angle. The two together written as above specify the normal k.

Reciprocal of a Vector.—By the reciprocal of a vector is meant the vector which combined with the original vector produces the product $+1$. The reciprocal of A is denoted by A^{-1}. Since $AB = ab\,(\cos \alpha\beta + \sin \alpha\beta \,.\, \overline{\alpha\beta})$, b must equal a^{-1} and β must be identical with α in order that the product may be 1. It follows that

$$A^{-1} = \frac{1}{a}\alpha = \frac{a\alpha}{a^2} = \frac{a_1 i + a_2 j}{a_1^2 + a_2^2}.$$

The reciprocal and opposite vector is $-A^{-1}$. In the figure let $\mathrm{OP} = 2\beta$ be the given vector; then $\mathrm{OQ} = \frac{1}{2}\beta$ is its reciprocal, and $\mathrm{OR} = \frac{1}{2}(-\beta)$ is its reciprocal and opposite.*

Example.—If $A = 10$ feet East $+ 5$ feet North, $A^{-1} = \frac{10}{125}$ feet East $+ \frac{5}{125}$ feet North and $-A^{-1} = \frac{-10}{125}$ feet East $- \frac{5}{125}$ feet North.

Product of the reciprocal of a vector and another vector.—

$$A^{-1}B = \frac{1}{a^2}AB,$$

$$= \frac{1}{a^2}\{a_1 b_1 + a_2 b_2 + (a_1 b_2 - a_2 b_1)\overline{\alpha\beta}\},$$

$$= \frac{b}{a}(\cos \alpha\beta + \sin \alpha\beta \,.\, \overline{\alpha\beta}).$$

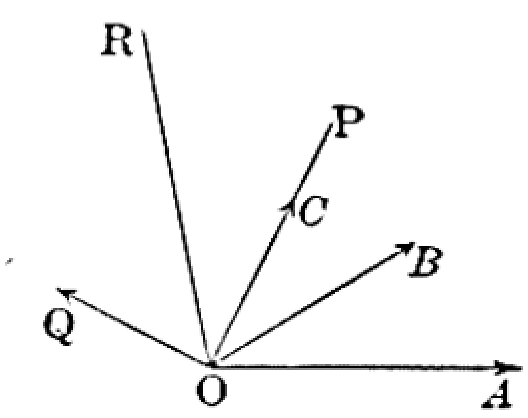
* Writers who identify a vector with a quadrantal versor are logically led to define the reciprocal of a vector as being opposite in direction as well as reciprocal in magnitude.

Hence $SA^{-1}B = \frac{b}{a}\cos \alpha\beta$ and $VA^{-1}B = \frac{b}{a}\sin \alpha\beta\,.\,\overline{\alpha\beta}$.

Product of three Coplanar Vectors.—Let $A = a_1 i + a_2 j$, $B = b_1 i + b_2 j$, $C = c_1 i + c_2 j$ denote any three vectors in a common plane. Then

$$\begin{aligned}(AB)C &= \{(a_1b_1 + a_2b_2) + (a_1b_2 - a_2b_1)k\}(c_1 i + c_2 j) \\ &= (a_1b_1 + a_2b_2)(c_1 i + c_2 j) + (a_1b_2 - a_2b_1)(-c_2 i + c_1 j).\end{aligned}$$

The former partial product means the vector C multiplied by the scalar product of A and B; while the latter partial product means the complementary vector of C multiplied by the magnitude of the vector product of A and B. If these partial products (represented by OP and OQ) unite to form a total product, the total product will be represented by OR, the resultant of OP and OQ.



The former product is also expressed by $SAB\,.\,C$, where the point separates the vectors to which the S refers; and more analytically by $abc \cos \alpha\beta\,.\,\gamma$.

The latter product is also expressed by $(VAB)C$, which is equivalent to $V(VAB)C$, because VAB is at right angles to C. It is also expressed by $abc \sin \alpha\beta\,.\,\overline{\overline{\alpha\beta}\gamma}$, where $\overline{\overline{\alpha\beta}\gamma}$ denotes the direction which is perpendicular to the perpendicular to α and β and γ.

If the product is formed after the other mode of association we have

$$\begin{aligned}A(BC) &= (a_1 i + a_2 j)(b_1c_1 + b_2c_2) + (a_1 i + a_2 j)(b_1c_2 - b_2c_1)k \\ &= (b_1c_1 + b_2c_2)(a_1 i + a_2 j) + (b_1c_2 - b_2c_1)(a_2 i - a_1 j) \\ &= SBC\,.\,A + VA(VBC).\end{aligned}$$

The vector $a_2 i - a_1 j$ is the opposite of the complementary vector of $a_1 i + a_2 j$. Hence the latter partial product differs with the mode of association.

Example.—Let $A = 1\underline{/0} + 2\underline{/90^\circ}$, $B = 3\underline{/0^\circ} + 4\underline{/90^\circ}$, $C = 5\underline{/0^\circ} + 6/90^\circ$. The fourth proportional to A, B, C is

$$(A^{-1}B)C = \frac{1 \times 3 + 2 \times 4}{1^2 + 2^2}\{5\underline{/0^\circ} + 6\underline{/90^\circ}\}$$
$$+ \frac{1 \times 4 - 2 \times 3}{1^2 + 2^2}\{-6\underline{/0^\circ} + 5\underline{/90^\circ}\}$$
$$= 13.4\underline{/0^\circ} + 11.2\underline{/90^\circ}.$$

Square of a Binomial of Vectors.—If $A + B$ denotes a sum of non-successive vectors, it is entirely equivalent to the resultant vector C. But the square of any vector is a positive scalar, hence the square of $A + B$ must be a positive scalar. Since A and B are in reality components of one vector, the square must be formed after the rules for the products of rectangular components (p. 432). Hence

$$\begin{aligned}(A + B)^2 &= (A + B)(A + B),\\ &= A^2 + AB + BA + B^2,\\ &= A^2 + B^2 + \mathrm{S}AB + \mathrm{S}BA + \mathrm{V}AB + \mathrm{V}BA,\\ &= A^2 + B^2 + 2\mathrm{S}AB.\end{aligned}$$

This may also be written in the form

$$a^2 + b^2 + 2ab \cos \alpha\beta.$$

But when $A + B$ denotes a sum of successive vectors, there is no third vector C which is the complete equivalent; and consequently we need not expect the square to be a scalar quantity. We observe that there is a real order, not of the factors, but of the terms in the binomial; this causes both product terms to be AB, giving

$$\begin{aligned}(A + B)^2 &= A^2 + 2AB + B^2\\ &= A^2 + B^2 + 2\mathrm{S}AB + 2\mathrm{V}AB.\end{aligned}$$

The scalar part gives the square of the length of the third side, while the vector part gives four times the area included between the path and the third side.

Square of a Trinomial of Coplanar Vectors.—Let $A + B + C$ denote a sum of successive vectors. The product terms must be formed so as to preserve the order of the vectors in the trinomial; that is, A is prior to B and C, and B is prior to C.

Hence

$$(A+B+C)^2 = A^2+B^2+C^2+2AB+2AC+2BC,$$
$$= A^2+B^2+C^2+2(\mathrm{S}AB+\mathrm{S}AC+\mathrm{S}BC), \quad (1)$$
$$+\,2(\mathrm{V}AB+\mathrm{V}AC+\mathrm{V}BC). \quad (2)$$

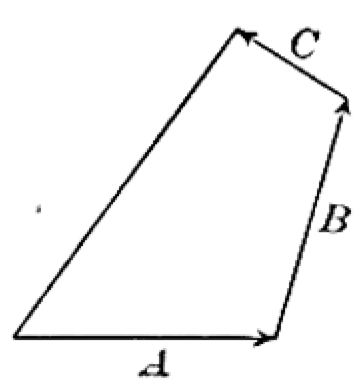
Hence $\mathrm{S}(A+B+C)^2 = (1)$

$$= a^2+b^2+c^2+2ab\cos\alpha\beta+2ac\cos\alpha\gamma+2bc\cos\beta\gamma$$

and $\mathrm{V}(A+B+C)^2 = (2)$

$$=\{2ab\sin\alpha\beta+2ac\sin\alpha\gamma+2bc\sin\beta\gamma\}.\overline{\alpha\beta}$$

The scalar part gives the square of the vector from the beginning of A to the end of C and is all that exists when the vectors are non-successive. The vector part is four times the area included between the successive sides and the resultant side of the polygon.



Note that it is here assumed that $\mathrm{V}(A+B)C = \mathrm{V}AC + \mathrm{V}BC$, which is the theorem of moments. Also that the product terms are not formed in cyclical order, but in accordance with the order of the vectors in the trinomial.

Example.—Let $A = 3\underline{/0}$, $B = 5\underline{/30^\circ}$, $C = 7\underline{/45^\circ}$; find the area of the polygon.

$$\tfrac{1}{2}\mathrm{V}(AB+AC+BC),$$
$$= \tfrac{1}{2}\{15\sin\underline{/0}\;\underline{/30^\circ}+21\sin\underline{/0}\;\underline{/45^\circ}+35\sin\underline{/30^\circ}\;\underline{/45^\circ}\},$$
$$= 3.75+7.42+4.53 = 15.7.$$

Prob. 10. At a distance of 25 centimeters $\underline{/20^\circ}$ there is a force of 1000 dynes $\underline{/80^\circ}$; find the moment.

Prob. 11. A conductor in an armature has a velocity of 240 inches per second $\underline{/300^\circ}$ and the magnetic flux is 50,000 lines per square inch $\underline{/0}$; find the vector product.

(Ans. 1.04×10^7 lines per inch per second.)

Prob. 12. Find the sine and cosine of the angle between the directions 0.8141 E. + 0.5807 N., and 0.5060 E. + 0.8625 N.

Prob. 13. When a force of 200 pounds $\underline{/270^\circ}$ is displaced by 10 feet $\underline{/30^\circ}$, what is the work done (scalar product)? What is the meaning of the negative sign in the scalar product?

Prob. 14. A mass of 100 pounds is moving with a velocity of 30 feet E. per second + 50 feet SE. per second; find its kinetic energy.

Prob. 15. A force of 10 pounds $\underline{/45^\circ}$ is acting at the end of 8 feet $\underline{/200^\circ}$; find the torque, or vector product.

Prob. 16. The radius of curvature of a curve is $2\underline{/0^\circ} + 5\underline{/90^\circ}$; find the curvature. (Ans. $.03\underline{/0^\circ} + .17\underline{/90^\circ}$.)

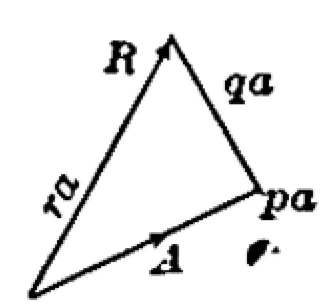
Prob. 17. Find the fourth proportional to $10\underline{/0^\circ} + 2\underline{/90^\circ}$ $8\underline{/0^\circ} - 3\underline{/90^\circ}$, and $6\underline{/0^\circ} + 5\underline{/90^\circ}$.

Prob. 18. Find the area of the polygon whose successive sides are $10\underline{/30^\circ}$, $9\underline{/100^\circ}$, $8\underline{/180^\circ}$, $7\underline{/225^\circ}$.

ART. 4. COAXIAL QUATERNIONS.

By a "quaternion" is meant the operator which changes one vector into another. It is composed of a magnitude and a turning factor. The magnitude may or may not be a mere ratio, that is, a quantity destitute of physical dimensions; for the two vectors may or may not be of the same physical kind. The turning is in a plane, that is to say, it is not conical. For the present all the vectors considered lie in a common plane; hence all the quaternions considered have a common axis.*

Let A and R be two coinitial vectors; the direction normal to the plane may be denoted by β. The operator which changes A into R consists of a scalar multiplier and a turning round the axis β. Let the former be denoted by r and the latter by β^θ, where θ denotes the angle in radians. Thus $R = r\beta^\theta A$ and reciprocally $A = \frac{1}{r}\beta^{-\theta}R$. Also $\frac{1}{A}R = r\beta^\theta$ and $\frac{1}{R}A = \frac{1}{r}\beta^{-\theta}$.



The turning factor β^θ may be expressed as the sum of two component operators, one of which has a zero angle and the other an angle of a quadrant. Thus

$$\beta^\theta = \cos\theta \,.\, \beta^0 + \sin\theta \,.\, \beta^{\pi/2}.$$

* The idea of the "quaternion" is due to Hamilton. Its importance may be judged from the fact that it has made solid trigonometrical analysis possible. It is the most important key to the extension of analysis to space. Etymologically "quaternion" means defined by four elements; which is true in space · in plane analysis it is defined by two.

When the angle is naught, the turning-factor may be omitted; but the above form shows that the equation is homogeneous, and expresses nothing but the equivalence of a given quaternion to two component quaternions.*

Hence
$$\begin{aligned} r\beta^{\theta} &= r\cos\theta + r\sin\theta\,.\,\beta^{\pi/2} \\ &= p + q\,.\,\beta^{\pi/2} \end{aligned}$$
and
$$\begin{aligned} r\beta^{\theta}A &= pA + q\beta^{\pi/2}A \\ &= pa\,.\,\alpha + qa\,.\,\beta^{\pi/2}\alpha. \end{aligned}$$

The relations between r and θ, and p and q, are given by
$$r = \sqrt{p^2+q^2}, \quad \theta = \tan^{-1}\frac{p}{q}.$$

Example.—Let E denote a sine alternating electromotive force in magnitude and phase, and I the alternating current in magitude and phase, then
$$E = (r + 2\pi nl\,.\,\beta^{\pi/2})I,$$
where r is the resistance, l the self-induction, n the alternations per unit of time, and β denotes the axis of the plane of representation. It follows that $E = rI + 2\pi nl\,.\,\beta^{\pi/2}I$; also that
$$I^{-1}E = r + 2\pi nl\,.\,\beta^{\pi/2},$$
that is, the operator which changes the current into the electromotive force is a quaternion. The resistance is the scalar part of the quaternion, and the inductance is the vector part.

Components of the Reciprocal of a Quaternion.—Given
$$R = (p + q\,.\,\beta^{\pi/2})A,$$
then
$$\begin{aligned} A &= \frac{1}{p+q\,.\,\beta^{\pi/2}}R \\ &= \frac{p - q\,.\,\beta^{\pi/2}}{(p+q\,.\,\beta^{\pi/2})(p-q\,.\,\beta^{\pi/2})}R \\ &= \frac{p-q\,.\,\beta^{\pi/2}}{p^2+q^2}R \\ &= \left\{\frac{p}{p^2+q^2} - \frac{q}{p^2+q^2}\,.\,\beta^{\pi/2}\right\}R. \end{aligned}$$

* In the method of complex numbers $\beta^{\pi/2}$ is expressed by i, which stands for $\sqrt{-1}$. The advantages of using the above notation are that it is capable of being applied to space, and that it also serves to specify the general turning factor β^{θ} as well as the quadrantal turning factor $\beta^{\pi/2}$.

Example.—Take the same application as above. It is important to obtain I in terms of E. By the above we deduce that from $E = (r + 2\pi nl \,.\, \beta^{\pi/2})I$

$$I = \left\{ \frac{r}{r^2 + (2\pi nl)^2} - \frac{2\pi nl}{r^2 + (2\pi nl)^2} . \beta^{\pi/2} \right\} E.$$

Addition of Coaxial Quaternions.—If the ratio of each of several vectors to a constant vector A is given, the ratio of their resultant to the same constant vector is obtained by taking the sum of the ratios. Thus, if

$$\begin{aligned} R_1 &= (p_1 + q_1 \,.\, \beta^{\pi/2})A, \\ R_2 &= (p_2 + q_2 \,.\, \beta^{\pi/2})A, \\ &\vdots \\ R_n &= (p_n + q_n \,.\, \beta^{\pi/2})A, \end{aligned}$$

then

$$\Sigma R = \{\Sigma p + (\Sigma q) . \beta^{\pi/2}\}A,$$

and reciprocally

$$A = \frac{\Sigma p - (\Sigma q) \,.\, \beta^{\pi/2}}{(\Sigma p)^2 + (\Sigma q)^2} \Sigma R.$$

Example.—In the case of a compound circuit composed of a number of simple circuits in parallel

$$I_1 = \frac{r_1 - 2\pi nl_1 . \beta^{\pi/2}}{r_1^2 + (2\pi n)^2 l_1^2} E, \qquad I_2 = \frac{r_2 - 2\pi nl_2 . \beta^{\pi/2}}{r_2^2 + (2\pi n)^2 l_2^2} E, \text{ etc.,}$$

therefore, $\Sigma I = \Sigma \left\{ \frac{r - 2\pi nl . \beta^{\pi/2}}{r^2 + (2\pi n)^2 l^2} \right\} E$

$$= \left\{ \Sigma\left(\frac{r}{r^2 + (2\pi n)^2 l^2}\right) - 2\pi n \Sigma \frac{l}{r^2 + (2\pi n)^2 l^2} . \beta^{\pi/2} \right\} E,$$

and reciprocally

$$E = \frac{\Sigma\left(\frac{r}{r^2 + (2\pi n)^2 l^2}\right) + 2\,\pi n \Sigma\left(\frac{l}{r^2 + (2\pi n)^2 l^2}\right) . \beta^{\pi/2}}{\left(\Sigma \frac{r}{r^2 + (2\pi n)^2 l^2}\right)^2 + (2\pi n)^2 \left(\Sigma \frac{l}{r^2 + (2\pi n)^2 l^2}\right)^2} \Sigma I.*$$

Product of Coaxial Quaternions.—If the quaternions which change A to R, and R to R', are given, the quaternion which changes A to R' is obtained by taking the product of the given quaternions.

* This theorem was discovered by Lord Rayleigh; Philosophical Magazine, May, 1886. See also Bedell & Crehore's Alternating Currents, p. 238.

Given $\quad R = r\beta^{\theta}A = (p + q \cdot \beta^{\pi/2})A$

and $\quad R' = r'\beta^{\theta'}R = (p' + q' \cdot \beta^{\pi/2})R,$

then $R' = rr'\beta^{\theta+\theta'} A = \{(pp' - qq') + (pq' + p'q) \cdot \beta^{\pi/2}\}A.$

Note that the product is formed by taking the product of the magnitudes, and likewise the product of the turning factors. The angles are summed because they are indices of the common base β.*

Quotient of two Coaxial Quaternions.—If the given quaternions are those which change A to R, and A to R', then that which changes R to R' is obtained by taking the quotient of the latter by the former.

Given $\quad R = r\beta^{\theta}A = (p + q \cdot \beta^{\pi/2})A$

and $\quad R' = r'\beta^{\theta'}A = (p' + q' \cdot \beta^{\pi/2})A,$

then
$$
\begin{aligned}
R' &= \frac{r'}{r}\beta^{\theta'-\theta}R, \\
&= (p' + q' \cdot \beta^{\pi/2})\frac{1}{p + q \cdot \beta^{\pi/2}}R, \\
&= (p' + q' \cdot \beta^{\pi/2})\frac{(p - q \cdot \beta^{\pi/2})}{p^2 + q^2}R, \\
&= \frac{(pp' + qq') + (pq' - p'q) \cdot \beta^{\pi/2}}{p^2 + q^2}R.
\end{aligned}
$$

Prob. 19. The impressed alternating electromotive force is 200 volts, the resistance of the circuit is 10 ohms, the self-induction is $\frac{1}{100}$ henry, and there are 60 alternations per second; required the current. (Ans. 18.7 amperes $\underline{/-20^\circ\ 42'}$.)

Prob. 20. If in the above circuit the current is 10 amperes, find the impressed voltage.

Prob. 21. If the electromotive force is 110 volts $\underline{/\theta}$ and the current is 10 amperes $\underline{/\theta - \frac{1}{4}\pi}$, find the resistance and the self-induction, there being 120 alternations per second.

Prob. 22. A number of coils having resistances r_1, r_2, etc., and self-inductions l_1, l_2, etc., are placed in series; find the impressed electromotive force in terms of the current, and reciprocally.

* Many writers, such as Hayward in "Vector Algebra and Trigonometry," and Stringham in "Uniplanar Algebra," treat this product of coaxial quaternions as if it were the product of vectors. This is the fundamental error in the Argand method.

ART. 5. ADDITION OF VECTORS IN SPACE.

A vector in space can be expressed in terms of three independent components, and when these form a rectangular set the directions of resolution are expressed by i, j, k. Any variable vector R may be expressed as $R = r\rho = xi + yj + zk$, and any constant vector B may be expressed as

$$B = b\beta = b_1 i + b_2 j + b_3 k.$$

In space the symbol ρ for the direction involves two elements. It may be specified as

$$\rho = \frac{xi + yj + zk}{x^2 + y^2 + z^2},$$

where the three squares are subject to the condition that their sum is unity. Or it may be specified by this notation, $\overline{\phi/}/\underline{\theta}$, a generalization of the notation for a plane. The additional angle $\overline{\phi/}$ is introduced to specify the plane in which the angle from the initial line lies.

If we are given R in the form $r\,\overline{\phi/}/\underline{\theta}$, then we deduce the other form thus:

$$R = r\cos\theta \,.\, i + r\sin\theta\cos\phi \,.\, j + r\sin\theta\sin\phi \,.\, k.$$

If R is given in the form $xi + yj + zk$, we deduce

$$R = \sqrt{x^2 + y^2 + z^2}\;\overline{\tan^{-1}\frac{z}{y}\Big/}\Big/\underline{\tan^{-1}\frac{\sqrt{y^2 + z^2}}{x}}.$$

For example, $B = 10\,\overline{30^\circ/}/\underline{45^\circ}$
$= 10\cos 45^\circ \,.\, i + 10\sin 45^\circ\cos 30^\circ \,.\, j + 10\sin 45^\circ\sin 30^\circ \,.\, k.$

Again, from $C = 3i + 4j + 5k$ we deduce

$$C = \sqrt{9 + 16 + 25}\;\overline{\tan^{-1}\frac{5}{4}\Big/}\Big/\underline{\tan^{-1}\frac{\sqrt{41}}{3}}$$

$$= 7.07\;\overline{51^\circ.4/}/\underline{64^\circ.9}.$$

To find the resultant of any number of component vectors applied at a common point, let $R_1, R_2, \ldots R_n$ represent the n vectors or,

$$R_1 = x_1 i + y_1 j + z_1 k,$$
$$R_2 = x_2 i + y_2 j + z_2 k,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$R_n = x_n i + y_n j + z_n k;$$

then
$$\Sigma R = (\Sigma x)i + (\Sigma y)j + (\Sigma z)k$$

and
$$r = \sqrt{(\Sigma x)^2 + (\Sigma y)^2 + (\Sigma z)^2},$$

$$\tan \phi = \frac{\Sigma z}{\Sigma y} \quad \text{and} \quad \tan \theta = \frac{\sqrt{(\Sigma y)^2 + (\Sigma z)^2}}{\Sigma x}.$$

Successive Addition.—When the successive vectors do not lie in one plane, the several elements of the area enclosed will lie in different planes, but these add by vector addition into a resultant directed area.

Prob. 23. Express $A = 4i - 5j + 6k$ and $B = 5i + 6j - 7k$ in the form $r\overline{\phi}/\!/\underline{\theta}$. (Ans. $8.8\ \overline{130^\circ}/\!/\underline{63^\circ}$ and $10.5\ \overline{311^\circ}/\!/\underline{61^\circ.5}$.)

Prob. 24. Express $C = 123\ \overline{57^\circ}/\!/\underline{142^\circ}$ and $D = 456\ \overline{65^\circ}/\!/\underline{200^\circ}$ in the form $xi + yj + zk$.

Prob. 25. Express $E = 100\ \overline{\frac{\pi}{4}}/\!/\underline{\frac{\pi}{3}}$ and $F = 1000\ \overline{\frac{\pi}{6}}/\!/\underline{3\frac{\pi}{4}}$ in the form $xi + yj + zk$.

Prob. 26. Find the resultant of $10\ \overline{20^\circ}/\!/\underline{30^\circ}$, $20\ \overline{30^\circ}/\!/\underline{40^\circ}$, and $30\ \overline{40^\circ}/\!/\underline{50^\circ}$.

Prob. 27. Express in the form $r\overline{\phi}/\!/\underline{\theta}$ the resultant vector of $1i + 2j - 3k$, $4i - 5j + 6k$, and $-7i + 8j + 9k$.

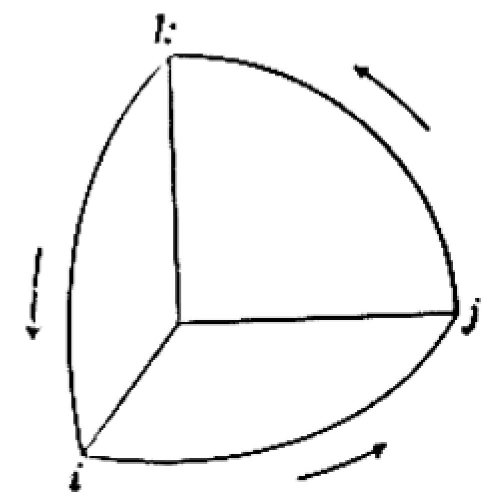
ART. 6. PRODUCT OF TWO VECTORS.

Rules of Signs for Vectors in Space.—By the rules $i^2 = +$, $j^2 = +$, $ij = k$, and $ji = -k$ we obtained (p. 432) a product of two vectors containing two partial products, each of which has the highest importance in mathematical and physical analysis. Accordingly, from the symmetry of space we assume that the following rules are true for the product of two vectors in space:

$$\begin{array}{lll} i^2 = +, & j^2 = +, & k^2 = +, \\ ij = k, & jk = i, & ki = j, \\ ji = -k, & kj = -i, & ik = -j. \end{array}$$

The square combinations give results which are indepen-

dent of direction, and consequently are summed by simple addition. The area vector determined by i and j can be represented in direction by k, because k is in tri-dimensional space the axis which is complementary to i and j. We also observe that the three rules $ij = k$, $jk = i$, $ki = j$ are derived from one another by cyclical permutation; likewise the three rules $ji = -k$, $kj = -i$, $ik = -j$. The figure shows that these rules are made to represent the relation of the advance to the rotation in the right-handed screw. The physical meaning of these rules is made clearer by an application to the dynamo and the electric motor. In the dynamo three principal vectors have to be considered: the velocity of the conductor at any instant, the intensity of magnetic flux, and the vector of electromotive force. Frequently all that is demanded is, given two of these directions to determine the third. Suppose that the direction of the velocity is i, and that of the flux j, then the direction of the electromotive force is k. The formula $ij = k$ becomes



velocity flux = electromotive-force,

from which we deduce

flux electromotive-force = velocity,

and electromotive-force velocity = flux.

The corresponding formula for the electric motor is

current flux = mechanical-force,

from which we derive by cyclical permutation

flux force = current, and force current = flux.

The formula velocity flux = electromotive-force is much handier than any thumb-and-finger rule; for it compares the three directions directly with the right-handed screw.

Example.—Suppose that the conductor is normal to the plane of the paper, that its velocity is towards the bottom, and that the magnetic flux is towards the left; corresponding to the rotation from the velocity to the flux in the right-handed screw we have advance into the paper: that then is the direction of the electromotive force.

Again, suppose that in a motor the direction of the current

along the conductor is up from the paper, and that the magnetic flux is to the left; corresponding to current flux we have advance towards the bottom of the page, which therefore must be the direction of the mechanical force which is applied to the conductor.

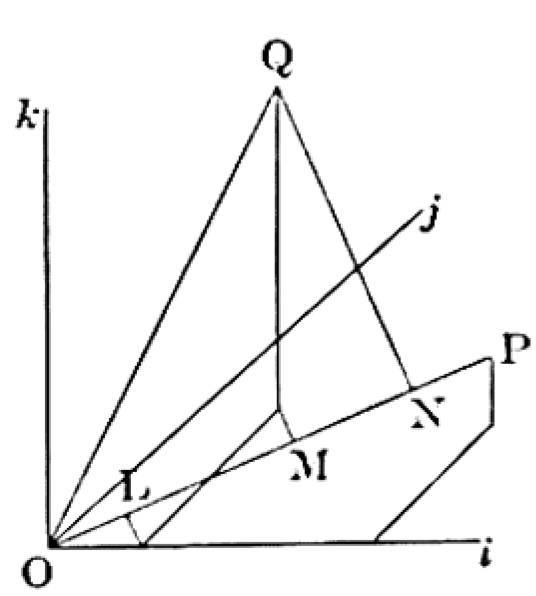
Complete Product of two Vectors.—Let $A = a_1 i + a_2 j + a_3 k$ and $B = b_1 i + b_2 j + b_3 k$ be any two vectors, not necessarily of the same kind physically, Their product, according to the rules (p. 444), is

$$\begin{aligned} AB &= (a_1 i + a_2 j + a_3 k)(b_1 i + b_2 j + b_3 k), \\ &= a_1 b_1 ii + a_2 b_2 jj + a_3 b_3 kk, \\ &+ a_2 b_3 jk + a_3 b_2 kj + a_3 b_1 ki + a_1 b_3 ik + a_1 b_2 ij + a_2 b_1 ji \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3, \\ &+ (a_2 b_3 - a_3 b_2) i + (a_3 b_1 - a_1 b_3) j + (a_1 b_2 - a_2 b_1) k \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3 + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ i & j & k \end{vmatrix}. \end{aligned}$$

Thus the product breaks up into two partial products, namely, $a_1 b_1 + a_2 b_2 + a_3 b_3$, which is independent of direction, and $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ i & j & k \end{vmatrix}$, which has the direction normal to the plane of A and B. The former is called the scalar product, and the latter the vector product.

In a sum of vectors, the vectors are necessarily homogeneous, but in a product the vectors may be heterogeneous. By making $a_3 = b_3 = 0$, we deduce the results already obtained for a plane.

Scalar Product of two Vectors.—The scalar product is denoted as before by SAB. Its geometrical meaning is the product of A and the orthogonal projection of B upon A. Let OP represent A, and OQ represent B, and let OL, LM, and MN be the orthogonal projections upon OP of the coordinates $b_1 i$, $b_2 j$, $b_3 k$ respectively. Then ON is the orthogonal projection of OQ and



$$\begin{aligned} \mathrm{OP} \times \mathrm{ON} &= \mathrm{OP} \times (\mathrm{OL} + \mathrm{LM} + \mathrm{MN}), \\ &= a\left(b_1\frac{a_1}{a} + b_2\frac{a_2}{a} + b_3\frac{a_3}{a}\right), \\ &= a_1b_1 + a_2b_2 + a_3b_3 = \mathrm{S}AB. \end{aligned}$$

Example.—Let the intensity of a magnetic flux be $B = b_1i + b_2j + b_3k$, and let the area be $S = s_1i + s_2j + s_3k$; then the flux through the area is $\mathrm{S}SB = b_1s_1 + b_2s_2 + b_3s_3$.

Corollary 1.—Hence $\mathrm{S}BA = \mathrm{S}AB$. For

$$b_1a_1 + b_2a_2 + b_3a_3 = a_1b_1 + a_2b_2 + a_3b_3.$$

The product of B and the orthogonal projection on it of A is equal to the product of A and the orthogonal projection on it of B. The product is positive when the vector and the projection have the same direction, and negative when they have opposite directions.

Corollary 2.—Hence $A^2 = a_1^2 + a_2^2 + a_3^2 = a^2$. The square of A must be positive; for the two factors have the same direction.

Vector Product of two Vectors.—The vector product as before is denoted by $\mathrm{V}AB$. It means the product of A and the component of B which is perpendicular to A, and is represented by the area of the parallelogram formed by A and B. The orthogonal projections of this area upon the planes of jk, ki, and ij represent the respective components of the product. For, let OP and OQ (see second figure of Art. 3) be the orthogonal projections of A and B on the plane of i and j; then the triangle OPQ is the projection of half of the parallelogram formed by A and B. But it is there shown that the area of the triangle OPQ is $\frac{1}{2}(a_1b_2 - a_2b_1)$. Thus $(a_1b_2 - a_2b_1)k$ denotes the magnitude and direction of the parallelogram formed by the projections of A and B on the plane of i and j. Similarly $(a_2b_3 - a_3b_2)i$ denotes in magnitude and direction the projection on the plane of j and k, and $(a_3b_1 - a_1b_3)j$ that on the plane of k and i.

Corollary 1.—Hence $\mathrm{V}BA = -\mathrm{V}AB$.

Example.—Given two lines $A = 7i - 10j + 3k$ and $B = -9i + 4j - 6k$; to find the rectangular projections of the parallelogram which they define:

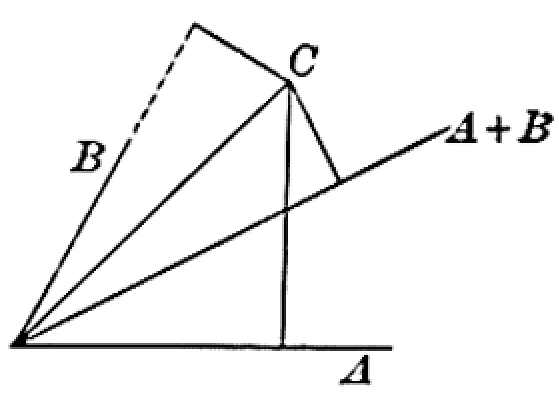
$$\begin{aligned} VAB &= (60 - 12)i + (-27 + 42)j + (28 - 90)k \\ &= 48i + 15j - 62k. \end{aligned}$$

Corollary 2.—If A is expressed as $a\alpha$ and B as $b\beta$, then $SAB = ab \cos \alpha\beta$ and $VAB = ab \sin \alpha\beta \,.\, \overline{\alpha\beta}$, where $\overline{\alpha\beta}$ denotes the direction which is normal to both α and β, and drawn in the sense given by the right-handed screw.

Example.—Given $A = r\overline{\phi}/\underline{/\theta}$ and $B = r'\overline{\phi'}/\underline{/\theta'}$. Then

$$\begin{aligned} SAB &= rr' \cos \overline{\phi}/\underline{/\theta}\ \overline{\phi'}/\underline{/\theta'} \\ &= rr'\{\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi' - \phi)\}. \end{aligned}$$

Product of two Sums of non-successive Vectors.—Let A and B be two component vectors, giving the resultant $A + B$, and let C denote any other vector having the same point of application.



Let
$$\begin{aligned} A &= a_1 i + a_2 j + a_3 k, \\ B &= b_1 i + b_2 j + b_3 k, \\ C &= c_1 i + c_2 j + c_3 k. \end{aligned}$$

Since A and B are independent of order,

$$A + B = (a_1 + b_1)i + (a_2 + b_2)j + (a_3 + b_3)k,$$

consequently by the principle already established

$$\begin{aligned} S(A + B)C &= (a_1 + b_1)c_1 + (a_2 + b_2)c_2 + (a_3 + b_3)c_3 \\ &= a_1c_1 + a_2c_2 + a_3c_3 + b_1c_1 + b_2c_2 + b_3c_3 \\ &= SAC + SBC. \end{aligned}$$

Similarly
$$\begin{aligned} V(A + B)C &= \{(a_2 + b_2)c_3 - (a_3 + b_3)c_2\}i + \text{etc.} \\ &= (a_2c_3 - a_3c_2)i + (b_2c_3 - b_3c_2)i + \ldots \\ &= VAC + VBC. \end{aligned}$$

Hence $$(A + B)C = AC + BC.$$

In the same way it may be shown that if the second factor consists of two components, C and D, which are non-successive in their nature, then

$$(A + B)(C + D) = AC + AD + BC + BD.$$

When $A+B$ is a sum of component vectors

$$\begin{aligned}(A+B)^2 &= A^2+B^2+AB+BA \\ &= A^2+B^2+2SAB.\end{aligned}$$

Prob. 28. The relative velocity of a conductor is S.W., and the magnetic flux is N.W.; what is the direction of the electromotive force in the conductor?

Prob. 29. The direction of the current is vertically downward, that of the magnetic flux is West; find the direction of the mechanical force on the conductor.

Prob. 30. A body to which a force of $2i+3j+4k$ pounds is applied moves with a velocity of $5i+6j+7k$ feet per second; find the rate at which work is done.

Prob. 31. A conductor $8i+9j+10k$ inches long is subject to an electromotive force of $11i+12j+13k$ volts per inch; find the difference of potential at the ends. (Ans. 326 volts.)

Prob. 32. Find the rectangular projections of the area of the parallelogram defined by the vectors $A = 12i-23j-34k$ and $B = -45i-56j+67k$.

Prob. 33. Show that the moment of the velocity of a body with respect to a point is equal to the sum of the moments of its component velocities with respect to the same point.

Prob. 34. The arm is $9i+11j+13k$ feet, and the force applied at either end is $17i+19j+23k$ pounds weight; find the torque.

Prob. 35. A body of 1000 pounds mass has linear velocities of 50 feet per second $\overline{30^\circ}/\underline{45^\circ}$, and 60 feet per second $\overline{60^\circ}/\underline{22^\circ.5}$; find its kinetic energy.

Prob. 36. Show that if a system of area-vectors can be represented by the faces of a polyhedron, their resultant vanishes.

Prob. 37. Show that work done by the resultant velocity is equal to the sum of the works done by its components.

Art. 7. Product of Three Vectors.

Complete Product.—Let us take $A = a_1i+a_2j+a_3k$, $B = b_1i+b_2j+b_3k$, and $C = c_1i+c_2j+c_3k$. By the product of A, B, and C is meant the product of the product of A and B with C, according to the rules p. 444). Hence

$$\begin{aligned}ABC &= (a_1b_1+a_2b_2+a_3b_3)(c_1i+c_2j+c_3k) \\ &+\{(a_2b_3-a_3b_2)i+(a_3b_1-a_1b_3)j+(a_1b_2-a_2b_1)k\}(c_1i+c_2j+c_3k) \\ &= (a_1b_1+a_2b_2+a_3b_3)(c_1i+c_2j+c_3k) \qquad (1)\end{aligned}$$

$$+\begin{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} & \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ c_1 & c_2 & c_3 \\ i & j & k \end{vmatrix} \quad (2) \qquad + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad (3)$$

Example.—Let $A = 1i + 2j + 3k$, $B = 4i + 5j + 6k$, and $C = 7i + 8j + 9k$. Then

$$(1) = (4 + 10 + 18)(7i + 8j + 9k) = 32(7i + 8j + 9k).$$

$$(2) = \begin{vmatrix} -3 & 6 & -3 \\ 7 & 8 & 9 \\ i & j & k \end{vmatrix} = 78i + 6j - 66k.$$

$$(3) = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0.$$

If we write $A = a\alpha$, $B = b\beta$, $C = c\gamma$, then

$$ABC = abc \cos \alpha\beta \,.\, \gamma \qquad (1)$$

$$+ abc \sin \alpha\beta \sin \overline{\overline{\alpha\beta}\gamma} \,.\, \overline{\overline{\overline{\alpha\beta}\gamma}} \qquad (2)$$

$$+ abc \sin \alpha\beta \cos \overline{\overline{\alpha\beta}\gamma}, \qquad (3)$$

where $\cos \overline{\overline{\alpha\beta}\gamma}$ denotes the cosine of the angle between the directions $\overline{\alpha\beta}$ and γ, and $\overline{\overline{\overline{\alpha\beta}\gamma}}$ denotes the direction which is normal to both $\overline{\alpha\beta}$ and γ.

We may also write

$$ABC = \underset{(1)}{SAB \,.\, C} + \underset{(2)}{V(VAB)C} + \underset{(3)}{S(VAB)C}.$$

First Partial Product.—It is merely the third vector multiplied by the scalar product of the other two, or weighted by that product as an ordinary algebraic quantity. If the directions are kept constant, each of the three partial products is proportional to each of the three magnitudes.

Second Partial Product.—The second partial product may be expressed as the difference of two products similar to the first. For

$$\begin{aligned} V(VAB)C = &\{-(b_2c_2 + b_3c_3)a_1 + (c_2a_2 + c_3a_3)b_1\}i \\ &+\{-(b_3c_3 + b_1c_1)a_2 + (c_3a_3 + c_1a_1)b_2\}j \\ &+\{-(b_1c_1 + b_2c_2)a_3 + (c_1a_1 + c_2a_2)b_3\}k. \end{aligned}$$

By adding to the first of these components the null term $(b_1 c_1 a_1 - c_1 a_1 b_1)i$ we get $-\mathrm{S}BC\,.\,a_1 i + \mathrm{S}CA\,.\,b_1 i$, and by treating the other two components similarly and adding the results we obtain

$$\mathrm{V}(\mathrm{V}AB)C = -\,\mathrm{S}BC\,.\,A + \mathrm{S}CA\,.\,B.$$

The principle here proved is of great use in solving equations (see p. 455).

Example.—Take the same three vectors as in the preceding example. Then

$$\begin{aligned}\mathrm{V}(\mathrm{V}AB)C &= -(28 + 40 + 54)(1i + 2j + 3k) \\ &\quad + (7 + 16 + 27)(4i + 5j + 6k) \\ &= 78i + 6j - 66k.\end{aligned}$$

The determinant expression for this partial product may also be written in the form

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\begin{vmatrix} c_1 & c_2 \\ i & j \end{vmatrix} + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\begin{vmatrix} c_2 & c_3 \\ j & k \end{vmatrix} + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}\begin{vmatrix} c_3 & c_1 \\ k & i \end{vmatrix}$$

It follows that the frequently occurring determinant expression

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\begin{vmatrix} c_2 & c_3 \\ d_2 & d_3 \end{vmatrix} + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}\begin{vmatrix} c_3 & c_1 \\ d_3 & d_1 \end{vmatrix}$$

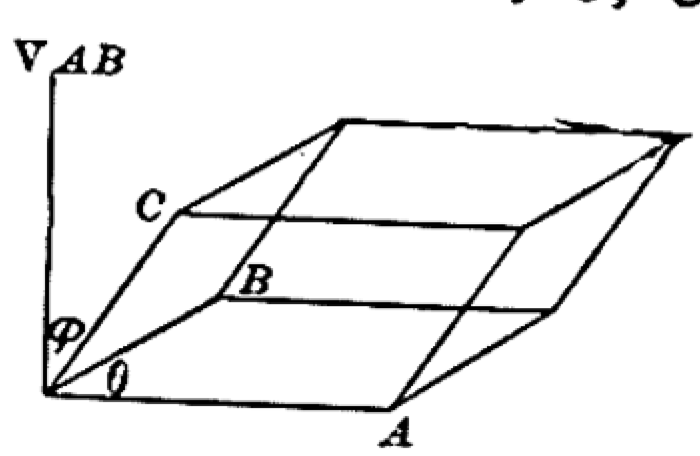
means $\mathrm{S}(\mathrm{V}AB)(\mathrm{V}CD)$.

Third Partial Product.—From the determinant expression for the third product, we know that

$$\begin{aligned}\mathrm{S}(\mathrm{V}AB)C &= \mathrm{S}(\mathrm{V}BC)A = \mathrm{S}(\mathrm{V}CA)B \\ &= -\,\mathrm{S}(\mathrm{V}BA)C = -\,\mathrm{S}(\mathrm{V}CB)A = -\,\mathrm{S}(\mathrm{V}AC)B.\end{aligned}$$

Hence any of the three former may be expressed by $\mathrm{S}ABC$, and any of the three latter by $-\,\mathrm{S}ABC$.

The third product $\mathrm{S}(\mathrm{V}AB)C$ is represented by the volume of the parallelepiped formed by the vectors A, B, C taken in that order. The line $\mathrm{V}AB$ represents in magnitude and direction the area formed by A and B, and the product of $\mathrm{V}AB$ with the projection of C upon it is the measure of the volume in magnitude and sign. Hence the volume formed by the three vectors has no direction in space, but it is positive or negative according to the cyclical order of the vectors.



In the expression $abc \sin \alpha\beta \cos \alpha\beta\gamma$ it is evident that $\sin \alpha\beta$ corresponds to $\sin \theta$, and $\cos \alpha\beta\gamma$ to $\cos \phi$, in the usual formula for the volume of a parallelepiped.

Example.—Let the velocity of a straight wire parallel to itself be $V = 1000 \underline{/30^\circ}$ centimeters per second, let the intensity of the magnetic flux be $B = 6000 \underline{/90^\circ}$ lines per square centimeter, and let the straight wire $L = 15$ centimeters $\overline{60^\circ/}\ \underline{/45^\circ}$. Then $\mathrm{V}VB = 6000000 \sin 60^\circ\, \overline{90^\circ/}\ \underline{/90^\circ}$ lines per centimeter per second. Hence $\mathrm{S}(\mathrm{V}VB)L = 15 \times 6000000 \sin 60^\circ \cos\phi$ lines per second where $\cos\phi = \sin 45^\circ \sin 60^\circ$.

Sum of the Partial Vector Products.—By adding the first and second partial products we obtain the total vector product of ABC, which is denoted by $\mathrm{V}(ABC)$. By decomposing the second product we obtain

$$\mathrm{V}(ABC) = \mathrm{S}AB.\, C - \mathrm{S}BC\,.\,A + \mathrm{S}CA\,.\,B.$$

By removing the common multiplier abc, we get

$$\mathrm{V}(\alpha\beta\gamma) = \cos \alpha\beta\,.\,\gamma - \cos \beta\gamma\,.\,\alpha + \cos \gamma\alpha\,.\,\beta.$$

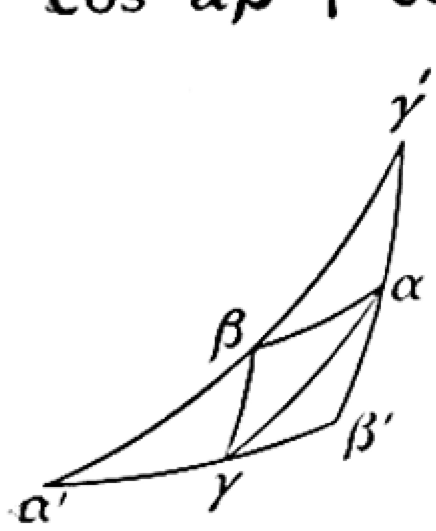
Similarly $\mathrm{V}(\beta\gamma\alpha) = \cos \beta\gamma\,.\,\alpha - \cos \gamma\alpha\,.\,\beta + \cos \alpha\beta\,.\,\gamma$

and $\mathrm{V}(\gamma\alpha\beta) = \cos \gamma\alpha\,.\,\beta - \cos \alpha\beta\,.\,\gamma + \cos \beta\gamma\,.\,\alpha.$

These three vectors have the same magnitude, for the square of each is

$$\cos^2 \alpha\beta + \cos^2 \beta\gamma + \cos^2 \gamma\alpha - 2 \cos \alpha\beta \cos \beta\gamma \cos \gamma\alpha,$$

that is, $1 - \{\mathrm{S}(\alpha\beta\gamma)\}^2$.



They have the directions respectively of α', β', γ', which are the corners of the triangle whose sides are bisected by the corners α, β, γ of the given triangle.

Prob. 38. Find the second partial product of $9\ \overline{20^\circ/}\,\underline{/30^\circ}$, $10\ \overline{30^\circ/}\,\underline{/40^\circ}$, $11\ \overline{45^\circ/}\,\underline{/45^\circ}$. Also the third partial product.

Prob. 39. Find the cosine of the angle between the plane of $l_1 i + m_1 j + n_1 k$ and $l_2 i + m_2 j + n_2 k$ and the plane of $l_3 i + m_3 j + n_3 k$ and $l_4 i + m_4 j + n_4 k$.

Prob. 40. Find the volume of the parallelepiped determined by the vectors $100i + 50j + 25k$, $50i + 10j + 80k$, and $-75i + 40j - 80k$.

Prob. 41. Find the volume of the tetrahedron determined by the extremities of the following vectors: $3i - 2j + 1k$, $-4i + 5j - 7k$, $3i - 7j - 2k$, $8i + 4j - 3k$.

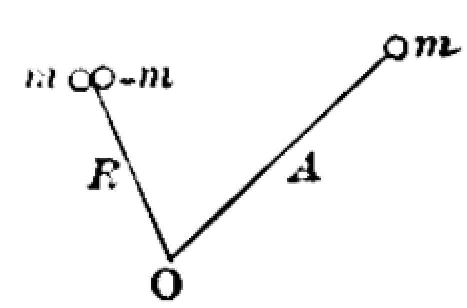
Prob. 42. Find the voltage at the terminals of a conductor when its velocity is 1500 centimeters per second, the intensity of the magnetic flux is 7000 lines per square centimeter, and the length of the conductor is 20 centimeters, the angle between the first and second being 30°, and that between the plane of the first two and the direction of the third 60°. (Ans. .91 volts.)

Prob. 43. Let $\alpha = \overline{20^\circ /}\underline{/10^\circ}$, $\beta = \overline{30^\circ /}\underline{/25^\circ}$, $\gamma = \overline{40^\circ /}\underline{/35^\circ}$. Find $V\alpha\beta\gamma$, and deduce $V\beta\gamma\alpha$ and $V\gamma\alpha\beta$.

Art. 8. Composition of Quantities.

A number of homogeneous quantities are simultaneously located at different points; it is required to find how to add or compound them.

Addition of a Located Scalar Quantity.—Let m_A denote a mass m situated at the extremity of the radius-vector A. A mass $m - m$ may be introduced at the extremity of any radius-vector R, so that



$$\begin{aligned} m_A &= (m - m)_R + m_A \\ &= m_R + m_A - m_R \\ &= m_R + m(A - R). \end{aligned}$$

Here $A - R$ is a simultaneous sum, and denotes the radius-vector from the extremity of R to the extremity of A. The product $m(A - R)$ is what Clerk Maxwell called a mass-vector, and means the directed moment of m with respect to the extremity of R. The equation states that the mass m at the extremity of the vector A is equivalent to the equal mass at the extremity of R, together with the said mass-vector applied at the extremity of R. The equation expresses a physical or mechanical principle.

Hence for any number of masses, m_1 at the extremity of A_1, m_2 at the extremity of A_2, etc.,

$$\Sigma m_A = \Sigma m_R + \Sigma\{m(A - R)\},$$

where the latter term denotes the sum of the mass-vectors treated as simultaneous vectors applied at a common point.

Since
$$\Sigma\{m(A-R)\} = \Sigma mA - \Sigma mR$$
$$= \Sigma mA - R\Sigma m,$$

the resultant moment will vanish if

$$R = \frac{\Sigma mA}{\Sigma m}, \quad \text{or} \quad R\Sigma m = \Sigma mA$$

Corollary.—Let $R = xi + yj + zk$,
and $A = a_1 i + b_1 j + c_1 k$;
then the above condition may be written as

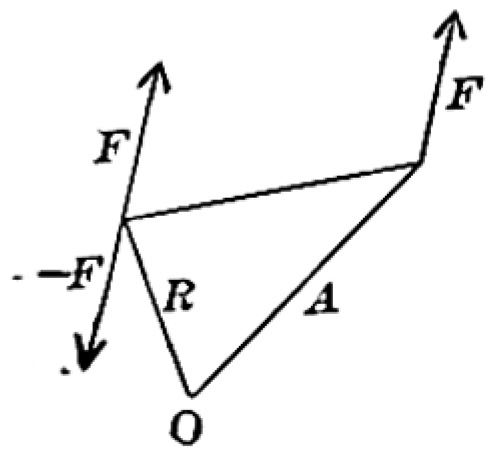
$$xi + yj + zk = \frac{\Sigma\{m(ai + bj + ck)\}}{\Sigma m}$$
$$= \frac{\Sigma(ma) \cdot i}{\Sigma m} + \frac{(\Sigma mb) \cdot j}{\Sigma m} + \frac{\Sigma(mc) \cdot k}{\Sigma m};$$

therefore
$$x = \frac{\Sigma(ma)}{\Sigma m}, \quad y = \frac{\Sigma(mb)}{\Sigma m}, \quad z = \frac{\Sigma mc}{\Sigma m}.$$

Example.—Given 5 pounds at 10 feet $\overline{45°}/\underline{/30°}$ and 8 pounds at 7 feet $\overline{60°}/\underline{/45°}$; find the moment when both masses are transferred to 12 feet $\overline{75°}/\underline{/60°}$.

$$m_1A_1 = 50(\cos 30° i + \sin 30° \cos 45° j + \sin 30° \sin 45° k),$$
$$m_2A_2 = 56(\cos 45° i + \sin 45° \cos 60° j + \sin 45° \sin 60° k),$$
$$(m_1 + m_2)R = 156(\cos 60° i + \sin 60° \cos 75° j + \sin 60° \sin 75° k),$$
$$\text{moment} = m_1A_1 + m_2A_2 - (m_1 + m_2)R.$$

Composition of a Located Vector Quantity.—Let F_A denote a force applied at the extremity of the radius-vector A.



As a force $F - F$ may introduced at the extremity of any radius-vector R, we have

$$F_A = (F - F)_R + F_A$$
$$= F_R + \mathrm{V}(A-R)F.$$

This equation asserts that a force F applied at the extremity of A is equivalent to an equal force applied at the extremity of R together with a couple whose magnitude

and direction are given by the vector product of the radius-vector from the extremity of R to the extremity of A and the force.

Hence for a system of forces applied at different points, such as F_1 at A_1, F_2 at A_2, etc., we obtain

$$\begin{aligned}\Sigma(F_A) &= \Sigma(F_R) + \Sigma \mathrm{V}(A-R)F \\ &= (\Sigma F)_R + \Sigma \mathrm{V}(A-R)F.\end{aligned}$$

Since
$$\begin{aligned}\Sigma \mathrm{V}(A-R)F &= \Sigma \mathrm{V}AF - \Sigma \mathrm{V}RF \\ &= \Sigma \mathrm{V}AF - \mathrm{V}R\Sigma F\end{aligned}$$

the condition for no resultant couple is

$$\mathrm{V}R\Sigma F = \Sigma \mathrm{V}AF,$$

which requires ΣF to be normal to $\Sigma \mathrm{V}AF$.

Example.—Given a force $1i+2j+3k$ pounds weight at $4i+5j+6k$ feet, and a force of $7i+9j+11k$ pounds weight at $10i+12j+14k$ feet; find the torque which must be supplied when both are transferred to $2i+5j+3k$, so that the effect may be the same as before.

$$\begin{aligned}\mathrm{V}A_1F_1 &= 3i-6j+3k, \\ \mathrm{V}A_2F_2 &= 6i-12j+6k, \\ \Sigma \mathrm{V}AF &= 9i-18j+9k, \\ \Sigma F &= 8i+11j+14k, \\ \mathrm{V}R\Sigma F &= 37i-4j-18k, \\ \text{Torque} &= -28i-14j+27k.\end{aligned}$$

By taking the vector product of the above equal vectors with the reciprocal of ΣF we obtain

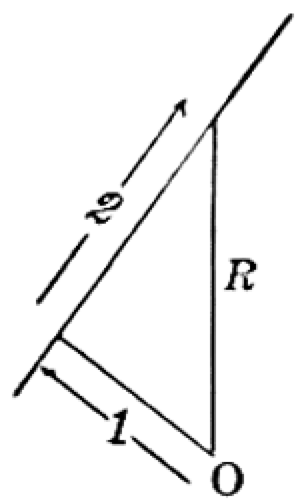
$$\mathrm{V}\left\{(\mathrm{V}R\Sigma F)\frac{1}{\Sigma F}\right\} = \mathrm{V}\left\{(\Sigma \mathrm{V}AF)\frac{1}{\Sigma F}\right\}.$$

By the principle previously established the left member resolves into $-R+\mathrm{S}R\frac{1}{\Sigma F}.\Sigma F$; and the right member is equivalent to the complete product on account of the two factors being normal to one another; hence

$$-R+\mathrm{S}R\frac{1}{\Sigma F}.\Sigma F = \Sigma(\mathrm{V}AF)\frac{1}{\Sigma F};$$

that is, $$R = \underset{(1)}{\frac{1}{\Sigma F}\Sigma(\mathrm{V}AF)} + \underset{(2)}{\mathrm{S}R\frac{1}{\Sigma F}\,.\,\Sigma F.}$$

The extremity of R lies on a straight line whose perpendicular is the vector (1) and whose direction is that of the resultant force. The term (2) means the projection of R upon that line.



The condition for the central axis is that the resultant force and the resultant couple should have the same direction; hence it is given by

$$\mathrm{V}\{\Sigma\mathrm{V}AF - \mathrm{V}R\Sigma F\}\Sigma F = 0;$$

that is, $$\mathrm{V}(\mathrm{V}R\Sigma F)\Sigma F = \mathrm{V}(\Sigma AF)\Sigma F.$$

By expanding the left member according to the same principle as above, we obtain

$$-(\Sigma F)^2 R + \mathrm{S}R\Sigma F\,.\,\Sigma F = \mathrm{V}(\Sigma AF)\Sigma F;$$

therefore $$R = \frac{1}{(\Sigma F)^2}\mathrm{V}\Sigma F(\mathrm{V}\Sigma AF) + \frac{\mathrm{S}R\Sigma F}{(\Sigma F)^2}\,.\,\Sigma F$$

$$= \mathrm{V}\left(\frac{1}{\Sigma F}\right)(\mathrm{V}\Sigma AF) + \mathrm{S}R\frac{1}{\Sigma F}\,.\,\Sigma F.$$

This is the same straight line as before, only no relation is now imposed on the directions of ΣF and $\Sigma\mathrm{V}AF$; hence there always is a central axis.

Example.—Find the central axis for the system of forces in the previous example. Since $\Sigma F = 8i + 11j + 14k$, the direction of the line is

$$\frac{8i + 11j + 14k}{\sqrt{64 + 121 + 196}}.$$

Since $\dfrac{1}{\Sigma F} = \dfrac{8i + 11j + 14k}{381}$ and $\Sigma\mathrm{V}AF = 9i - 18j + 9k$, the perpendicular to the line is

$$\mathrm{V}\frac{8i + 11j + 14k}{381}9i - 18j + 9k = \frac{1}{381}\{351i + 54j - 243k\}.$$

Prob. 44. Find the moment at $\overline{90^\circ}/\underline{270^\circ}$ of 10 pounds at 4 feet $\overline{10^\circ}/\underline{20^\circ}$ and 20 pounds at 5 feet $\overline{30^\circ}/\underline{120^\circ}$.

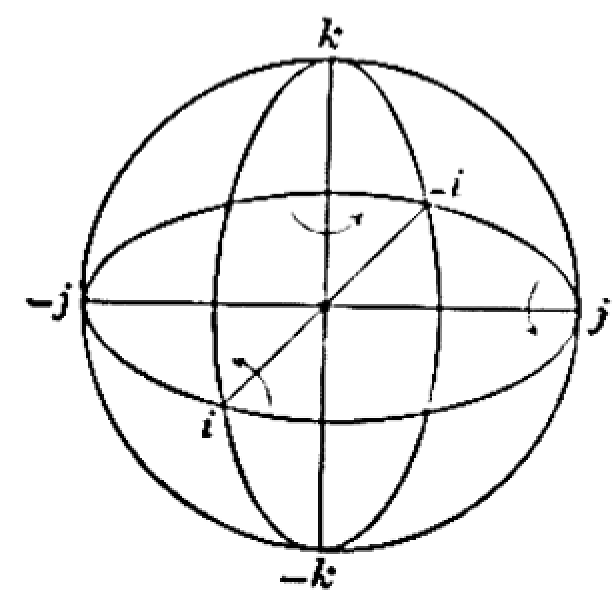
Prob. 45. Find the torque for $4i + 3j + 2k$ pounds weight at $2i - 3j + 1k$ feet, and $2i - 1j - 1k$ pounds weight at $-3i + 4j + 5k$ feet when transferred to $-3i + 2j - 4k$ feet.

Prob. 46. Find the central axis in the above case.

Prob. 47. Prove that the mass-vector drawn from any origin to a mass equal to that of the whole system placed at the center of mass of the system is equal to the sum of the mass-vectors drawn from the same origin to all the particles of the system.

Art. 9. Spherical Trigonometry.

Let i, j, k denote three mutually perpendicular axes. In order to distinguish clearly between an axis and a quadrantal version round it, let $i^{\pi/2}, j^{\pi/2}, k^{\pi/2}$ denote quadrantal versions in the positive sense about the axes i, j, k respectively. The directions of positive version are indicated by the arrows.



By $i^{\pi/2}i^{\pi/2}$ is meant the product of two quadrantal versions round i; it is equivalent to a semicircular version round i; hence $i^{\pi/2}i^{\pi/2} = i^{\pi} = -$. Similarly $j^{\pi/2}j^{\pi/2}$ means the product of two quadrantal versions round j, and $j^{\pi/2}j^{\pi/2} = j^{\pi} = -$. Similarly $k^{\pi/2}k^{\pi/2} = k^{\pi} = -$.

By $i^{\pi/2}j^{\pi/2}$ is meant a quadrant round i followed by a quadrant round j; it is equivalent to the quadrant from j to i, that is, to $-k^{\pi/2}$. But $j^{\pi/2}i^{\pi/2}$ is equivalent to the quadrant from $-i$ to $-j$, that is, to $k^{\pi/2}$. Similarly for the other two pairs of products. Hence we obtain the following

Rules for Versors.

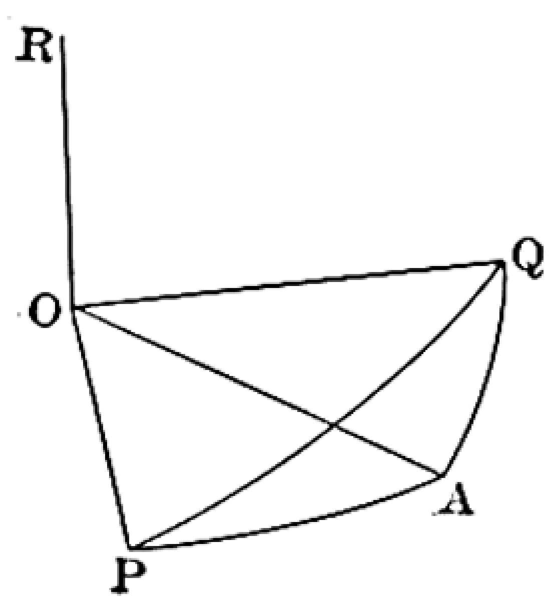
$$i^{\pi/2}i^{\pi/2} = -, \qquad j^{\pi/2}j^{\pi/2} = -, \qquad k^{\pi/2}k^{\pi/2} = -,$$
$$i^{\pi/2}j^{\pi/2} = -k^{\pi/2}, \qquad j^{\pi/2}i^{\pi/2} = k^{\pi/2},$$
$$j^{\pi/2}k^{\pi/2} = -i^{\pi/2}, \qquad k^{\pi/2}j^{\pi/2} = i^{\pi/2},$$
$$k^{\pi/2}i^{\pi/2} = -j^{\pi/2}, \qquad i^{\pi/2}k^{-/2} = j^{\pi/2}.$$

The meaning of these rules will be seen from the following application. Let $li + mj + nk$ denote any axis, then

$(li+mj+nk)^{\pi/2}$ denotes a quadrant of angle round that axis. This quadrantal version can be decomposed into the three rectangular components $li^{\pi/2}$, $mj^{\pi/2}$, $nk^{\pi/2}$; and these components are not successive versions, but the parts of one version. Similarly any other quadrantal version $(l'i+m'j+n'k)^{\pi/2}$ can be resolved into $l'i^{\pi/2}$, $m'j^{\pi/2}$, $n'k^{\pi/2}$. By applying the above rules, we obtain

$$\begin{aligned}(li+mj+nk)^{\pi/2}&(l'i+m'j+n'k)^{\pi/2}\\ &= (li^{\pi/2}+mj^{\pi/2}+nk^{\pi/2})(l'i^{\pi/2}+m'j^{\pi/2}+n'k^{\pi/2})\\ &= -(ll'+mm'+nn')\\ &\quad -(mn'-m'n)i^{\pi/2}-(nl'-n'l)j^{\pi/2}-(lm'-l'm)k^{\pi/2}\\ &= -(ll'+mm'+nn')\\ &\quad -\{(mn'-m'n)i+(nl'-n'l)j+(lm'-l'm)k\}^{\pi/2}.\end{aligned}$$

Product of Two Spherical Versors.—Let β denote the axis and b the ratio of the spherical versor PA, then the versor itself is expressed by β^b. Similarly let γ denote the axis and c the ratio of the spherical versor AQ, then the versor itself is expressed by γ^c.



Now $\beta^b = \cos b + \sin b\,.\,\beta^{\pi/2}$,

and $\gamma^c = \cos c + \sin c\,.\,\gamma^{\pi/2}$;

therefore

$$\begin{aligned}\beta^b\gamma^c &= (\cos b+\sin b\,.\,\beta^{\pi/2})(\cos c+\sin c\,.\,\gamma^{\pi/2})\\ &= \cos b\cos c+\cos b\sin c\,.\,\gamma^{\pi/2}+\cos c\sin b\,.\,\beta^{\pi/2}\\ &\qquad +\sin b\sin c\,.\,\beta^{\pi/2}\gamma^{\pi/2}.\end{aligned}$$

But from the preceding paragraph

$$\beta^{\pi/2}\gamma^{\pi/2} = -\cos\beta\gamma - \sin\beta\gamma\,.\,\overline{\beta\gamma}^{\pi/2};$$

therefore $\beta^b\gamma^c = \cos b\cos c - \sin b\sin c\cos\beta\gamma$ (1)

$+\{\cos b\sin c\,.\,\gamma+\cos c\sin b\,.\,\beta-\sin b\sin c\sin\beta\gamma\,.\,\overline{\beta\gamma}\}^{\pi/2}$. (2)

The first term gives the cosine of the product versor; it is equivalent to the fundamental theorem of spherical trigonometry, namely,

$$\cos a = \cos b\cos c + \sin b\sin c\cos A,$$

where A denotes the external angle instead of the angle included by the sides.

The second term is the directed sine of the angle; for the square of (2) is equal to 1 minus the square of (1), and its direction is normal to the plane of the product angle.*

Example.—Let $\beta = \overline{30°}/\underline{45°}$ and $\gamma = \overline{60°}/\underline{30°}$. Then

$$\cos \beta\gamma = \cos 45° \cos 30° + \sin 45° \sin 30° \cos 30°,$$

and $$\sin \beta\gamma \,.\, \overline{\beta\gamma} = \mathrm{V}\beta\gamma;$$

but $\beta = \cos 45°\, i + \sin 45° \cos 30°\, j + \sin 45° \sin 30°\, k,$

and $\gamma = \cos 30°\, i + \sin 30° \cos 60°\, j + \sin 30° \sin 60°\, k;$

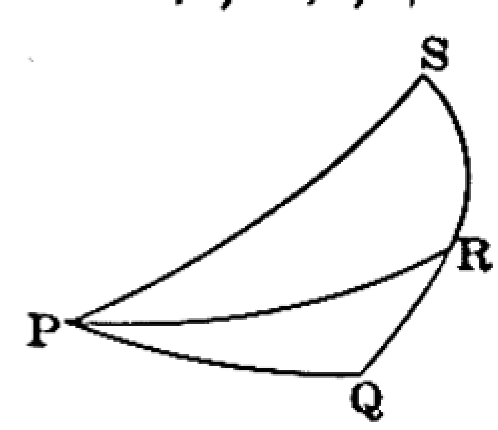
therefore

$$\begin{aligned}\mathrm{V}\beta\gamma = \{&\sin 45° \cos 30° \sin 30° \sin 60° \\ &- \sin 45° \sin 30° \sin 30° \cos 60°\}\, i \\ &+ \{\sin 45° \sin 30° \cos 30° - \cos 45° \sin 30° \sin 60°\}\, j \\ &+ \{\cos 45° \sin 30° \cos 60° - \sin 45° \cos 30° \cos 30°\}\, k.\end{aligned}$$

Quotient of Two Spherical Versors.—The reciprocal of a given versor is derived by changing the sign of the index; γ^{-c} is the reciprocal of γ^{c}. As $\beta^{b} = \cos b + \sin b \,.\, \beta^{\pi/2}$, and $\gamma^{-c} = \cos c - \sin c \,.\, \gamma^{\pi/2}$,

$$\begin{aligned}\beta^{b}\gamma^{-c} &= \cos b \cos c + \sin b \sin c \cos \beta\gamma \\ &+ \{\cos c \sin b \,.\, \beta - \cos b \sin c \,.\, \gamma + \sin b \sin c \sin \beta\gamma \,.\, \overline{\beta\gamma}\}^{\pi/2}.\end{aligned}$$

Product of Three Spherical Versors.—Let α^{a} denote the versor PQ, β^{b} the versor QR, and γ^{c} the versor RS; then $\alpha^{a}\beta^{b}\gamma^{c}$ denotes PS. Now $\alpha^{a}\beta^{b}\gamma^{c}$



$$\begin{aligned}&= (\cos a + \sin a \,.\, \alpha^{\pi/2})(\cos b + \sin b \,.\, \beta^{\pi/2})(\cos c + \sin c \,.\, \gamma^{\pi/2}) \\ &= \cos a \cos b \cos c \qquad (1) \\ &\quad + \cos a \cos b \sin c \,.\, \gamma^{\pi/2} + \cos a \cos c \sin b \,.\, \beta^{\pi/2} \\ &\quad + \cos b \cos c \sin a \,.\, \alpha^{\pi/2} \qquad (2) \\ &\quad + \cos a \sin b \sin c \,.\, \beta^{\pi/2}\gamma^{\pi/2} + \cos b \sin a \sin c \,.\, \alpha^{\pi/2}\gamma^{\pi/2} \\ &\quad + \cos c \sin a \sin b \,.\, \alpha^{\pi/2}\beta^{\pi/2} \qquad (3)\end{aligned}$$

* Principles of Elliptic and Hyperbolic Analysis, p. 2.

$$+ \sin a \sin b \sin c \,.\, \alpha^{\pi/2}\beta^{\pi/2}\gamma^{\pi/2}. \tag{4}$$

The versors in (3) are expanded by the rule already obtained, namely,

$$\beta^{\pi/2}\gamma^{\pi/2} = -\cos\beta\gamma - \sin\beta\gamma \,.\, \overline{\beta\gamma}^{\pi/2}.$$

The versor of the fourth term is

$$\alpha^{\pi/2}\beta^{\pi/2}\gamma^{\pi/2} = -\left(\cos\alpha\beta + \sin\alpha\beta \,.\, \overline{\alpha\beta}^{\pi/2}\right)\gamma^{\pi/2}$$

$$= -\cos\alpha\beta \,.\, \gamma^{\pi/2} + \sin\alpha\beta\cos\overline{\alpha\beta}\gamma + \sin\alpha\beta\sin\overline{\alpha\beta}\gamma \,.\, \overline{\overline{\alpha\beta}\gamma}^{\pi/2}.$$

Now $\sin\alpha\beta\sin\overline{\alpha\beta}\gamma \,.\, \overline{\overline{\alpha\beta}\gamma} = \cos\alpha\gamma \,.\, \beta - \cos\beta\gamma \,.\, \alpha$ (p. 451), hence the last term of the product, when expanded, is

$$\sin a \sin b \sin c\{-\cos\alpha\beta \,.\, \gamma^{\pi/2} + \cos\alpha\gamma \,.\, \beta^{\pi/2} - \cos\beta\gamma \,.\, \alpha^{\pi/2} + \cos\overline{\alpha\beta}\gamma\}.$$

Hence

$$\begin{aligned}\cos\alpha^a\beta^b\gamma^c = {} & \cos a\cos b\cos c - \cos a\sin b\sin c\cos\beta\gamma \\ & - \cos b\sin a\sin c\cos\alpha\gamma - \cos c\sin a\sin b\cos\alpha\beta \\ & + \sin a\sin b\sin c\sin\alpha\beta\cos\overline{\alpha\beta}\gamma,\end{aligned}$$

and, letting Sin denote the directed sine,

$$\begin{aligned}\operatorname{Sin}\alpha^a\beta^b\gamma^c = {} & \cos a\cos b\sin c \,.\, \gamma + \cos a\cos c\sin b \,.\, \beta \\ & + \cos b\cos c\sin a \,.\, \alpha - \cos a\sin b\sin c\sin\beta\gamma \,.\, \overline{\beta\gamma} \\ & - \cos b\sin a\sin c\sin\alpha\gamma \,.\, \overline{\alpha\gamma} \\ & - \cos c\sin a\sin b\sin\alpha\beta \,.\, \overline{\alpha\beta} \\ & - \sin a\sin b\sin c\{\cos\alpha\beta \,.\, \gamma - \cos\alpha\gamma \,.\, \beta + \cos\beta\gamma \,.\, \alpha\}.^*\end{aligned}$$

Extension of the Exponential Theorem to Spherical Trigonometry.—It has been shown (p. 458) that

$$\cos\beta^b\gamma^c = \cos b\cos c - \sin b\sin c\cos\beta\gamma$$

and

$$(\sin\beta^b\gamma^c)^{\pi/2} = \cos c\sin b \,.\, \beta^{\pi/2} + \cos b\sin c \,.\, \gamma^{\pi/2} - \sin b\sin c\sin\beta\gamma \,.\, \overline{\beta\gamma}^{\pi/2}.$$

Now $$\cos b = 1 - \frac{b^2}{2!} + \frac{b^4}{4!} - \frac{b^6}{6!} + \text{etc.}$$

* In the above case the three axes of the successive angles are not perfectly independent, for the third angle must begin where the second leaves off. But the theorem remains true when the axes are independent; the factors are then quaternions in the most general sense.

and $$\sin b = b - \frac{b^3}{3!} + \frac{b^5}{5!} - \text{etc.}$$

Substitute these series for cos b, sin b, cos c, and sin c in the above equations, multiply out, and group the homogeneous terms together. It will be found that

$$\cos \beta^b\gamma^c = 1 - \frac{1}{2!}\{b^2 + 2bc \cos \beta\gamma + c^2\}$$
$$+ \frac{1}{4!}\{b^4 + 4b^3c \cos \beta\gamma + 6b^2c^2 + 4bc^3 \cos \beta\gamma + c^4\}$$
$$- \frac{1}{6!}\{b^6 + 6b^5c \cos \beta\gamma + 15b^4c^2 + 20b^3c^3 \cos \beta\gamma$$
$$+ 15b^2c^4 + 6bc^5 \cos \beta\gamma + c^6\} + \ldots,$$

where the coefficients are those of the binomial theorem, the only difference being that cos $\beta\gamma$ occurs in all the odd terms as a factor. Similarly, by expanding the terms of the sine, we obtain

$$(\operatorname{Sin} \beta^b\gamma^c)^{\pi/2} = b \,.\, \beta^{\pi/2} + c \,.\, \gamma^{\pi/2} - bc \sin \beta\gamma \,.\, \overline{\beta\gamma}^{\pi/2}$$
$$- \frac{1}{3!}\{b^3 \,.\, \beta^{\pi/2} + 3b^2c \,.\, \gamma^{\pi/2} + 3bc^2 \,.\, \beta^{\pi/2} + c^3 \,.\, \gamma^{\pi/2}\}$$
$$+ \frac{1}{3!}\{bc^3 + b^3c\} \sin \beta\gamma \,.\, \overline{\beta\gamma}^{\pi/2}$$
$$+ \frac{1}{5!}\{b^5 \,.\, \beta^{\pi/2} + 5b^4c \,.\, \gamma^{\pi/2} + 10b^3c^2 \,.\, \beta^{\pi/2}$$
$$+ 10b^2c^3 \,.\, \gamma^{\pi/2} + 5bc^4 \,.\, \beta^{\pi/2} + c^5 \,.\, \gamma^{\pi/2}\}$$
$$- \frac{1}{5!}\left\{b^5c + \frac{5 \cdot 4}{2 \,.\, 3}b^3c^3 + bc^5\right\} \sin \beta\gamma \,.\, \overline{\beta\gamma}^{\pi/2} - \ldots$$

By adding these two expansions together we get the expansion for $\beta^b\gamma^c$, namely,

$$\beta^b\gamma^c = 1 + b \,.\, \beta^{\pi/2} + c \,.\, \gamma^{\pi/2}$$
$$- \frac{1}{2!}\{b^2 + 2bc(\cos \beta\gamma + \sin \beta\gamma \,.\, \overline{\beta\gamma}^{\pi/2}) + c^2\}$$
$$- \frac{1}{3!}\{b^3 \,.\, \beta^{\pi/2} + 3b^2c \,.\, \gamma^{\pi/2} + 3bc^2 \,.\, \beta^{\pi/2} + c^3 \,.\, \gamma^{\pi/2}\}$$
$$+ \frac{1}{4!}\{b^4 + 4b^3c(\cos \beta\gamma + \sin \beta\gamma \,.\, \overline{\beta\gamma}^{\pi/2}) + 6b^2c^2$$
$$+ 4bc^3(\cos \beta\gamma + \sin \beta\gamma \,.\, \overline{\beta\gamma}^{\pi/2}) + c^4\} + \ldots$$

By restoring the minus, we find that the terms on the second line can be thrown into the form

$$\frac{1}{2!}\{b^2 . \beta^{\pi} + 2bc . \beta^{\pi/2}\gamma^{\pi/2} + c^2 . \gamma^{\pi}\},$$

and this is equal to

$$\frac{1}{2!}\{b . \beta^{\pi/2} + c . \gamma^{\pi/2}\}^2,$$

where we have the square of a sum of successive terms. In a similar manner the terms on the third line can be restored to

$$b^3 . \beta^{3\pi/2} + 3b^2c . \beta^{\pi}\gamma^{\pi/2} + 3bc^2 . \beta^{\pi/2}\gamma^{\pi} + c^3 . \gamma^{3(\pi/2)},$$

that is, $$\frac{1}{3!}\{b . \beta^{\pi/2} + c . \gamma^{\pi/2}\}^3.$$

Hence

$$\beta^b\gamma^c = 1 + b . \beta^{\pi/2} + c . \gamma^{\pi/2} + \frac{1}{2!}\{b . \beta^{\pi/2} + c . \gamma^{\pi/2}\}^2$$
$$+ \frac{1}{3!}\{b . \beta^{\pi/2} + c . \gamma^{\pi/2}\}^3 + \frac{1}{4!}\{b . \beta^{\pi/2} + c . \gamma^{\pi/2}\}^4 +$$
$$= e^{b . \beta^{\pi/2} + c . \gamma^{\pi/2}}.*$$

Extension of the Binomial Theorem.—We have proved above that $e^{b\beta^{\pi/2}}e^{c\gamma^{\pi/2}} = e^{b\beta^{\pi/2} + c\gamma^{\pi/2}}$ provided that the powers of the binomial are expanded as due to a successive sum, that is, the order of the terms in the binomial must be preserved. Hence the expansion for a power of a successive binomial is given by

$$\{b . \beta^{\pi/2} + c . \gamma^{\pi/2}\}^n = b^n . \beta^{n\pi/2} + nb^{n-1}c . \beta^{(n-1)(\pi/2)}\gamma^{\pi/2}$$
$$+ \frac{n(n-1)}{1 . 2}b^{n-2}c^2 . \beta^{(n-2)(\pi/2)}\gamma^{\pi} + \text{etc.}$$

* At page 386 of his Elements of Quaternions, Hamilton says: "In the present theory of diplanar quaternions we cannot expect to find that the sum of the logarithms of any two proposed factors shall be generally equal to the logarithm of the product; but for the simpler and earlier case of coplanar quaternions, that algebraic property may be considered to exist, with due modification for multiplicity of value." He was led to this view by not distinguishing between vectors and quadrantal quaternions and between simultaneous and successive addition. The above demonstration was first given in my paper on "The Fundamental Theorems of Analysis generalized for Space." It forms the key to the higher development of space analysis.

Example.—Let $b = \frac{1}{10}$ and $c = \frac{1}{5}$, $\beta = \overline{30^\circ}/\underline{45^\circ}$, $\gamma = \overline{60^\circ}/\underline{30^\circ}$.

$$(b \, . \, \beta^{\pi/2} + c \, . \, \gamma^{\pi/2})^2 = -\{b^2 + c^2 + 2bc \cos \beta\gamma + 2bc(\sin \beta\gamma)^{\pi/2}\}$$
$$= -(\tfrac{1}{100} + \tfrac{1}{25} + \tfrac{2}{50} \cos \beta\gamma) - \tfrac{2}{50}(\sin \beta\gamma)^{\pi/2}.$$

Substitute the calculated values of $\cos \beta\gamma$ and $\sin \beta\gamma$ (page 459).

Prob. 48. Find the equivalent of a quadrantal version round $\frac{\sqrt{3}}{2}i + \frac{1}{2\sqrt{2}}j + \frac{1}{2\sqrt{2}}k$ followed by a quadrantal version round $\frac{1}{2}i + \frac{\sqrt{3}}{4}j + \frac{3}{4}k$.

Prob. 49. In the example on p. 459 let $b = 25^\circ$ and $c = 50^\circ$; calculate out the cosine and the directed sine of the product angle.

Prob. 50. In the above example calculate the cosine and the directed sine up to and inclusive of the fourth power of the binomial. (Ans. $\cos = .9735$.)

Prob. 51. Calculate the first four terms of the series when $b = \frac{1}{50}$, $c = \frac{1}{100}$, $\beta = \overline{0}/\underline{0}$, $\gamma = \overline{90^\circ}/\underline{90^\circ}$.

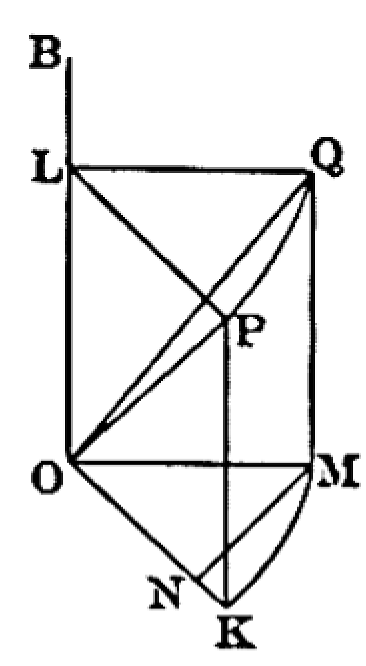
Prob. 52. From the fundamental theorem of spherical trigonometry deduce the polar theorem with respect to both the cosine and the directed sine.

Prob. 53. Prove that if α^a, β^b, γ^c denote the three versors of a spherical triangle, then

$$\frac{\sin \beta\gamma}{\sin a} = \frac{\sin \gamma\alpha}{\sin b} = \frac{\sin \alpha\beta}{\sin c}.$$

ART. 10. COMPOSITION OF ROTATIONS.

A version refers to the change of direction of a line, but a rotation refers to a rigid body. The composition of rotations is a different matter from the composition of versions.



Effect of a Finite Rotation on a Line.—Suppose that a rigid body rotates θ radians round the axis β passing through the point O, and that R is the radius-vector from O to some particle. In the diagram OB represents the axis β, and OP the vector R. Draw OK and OL, the rectangular components of R.

$$\beta^\theta R = (\cos \theta + \sin \theta \, . \, \beta^{\pi/2}) r\rho$$

$$= r(\cos\theta + \sin\theta \,.\, \beta^{\pi/2})(\cos\beta\rho \,.\, \beta + \sin\beta\rho \,.\, \overline{\overline{\beta\rho}\beta})$$
$$= r\{\cos\beta\rho \,.\, \beta + \cos\theta\sin\beta\rho \,.\, \overline{\overline{\beta\rho}\beta} + \sin\theta\sin\beta\rho \,.\, \overline{\beta\rho}\}.$$

When $\cos\beta\rho = 0$, this reduces to

$$\beta^\theta R = \cos\theta R + \sin\theta \mathrm{V}(\beta R).$$

The general result may be written

$$\beta^\theta R = \mathrm{S}\beta R \,.\, \beta + \cos\theta(\mathrm{V}\beta R)\beta + \sin\theta\mathrm{V}\beta R.$$

Note that $(\mathrm{V}\beta R)\beta$ is equal to $\mathrm{V}(\mathrm{V}\beta R)\beta$ because $\mathrm{S}\beta R\beta$ is 0, for it involves two coincident directions.

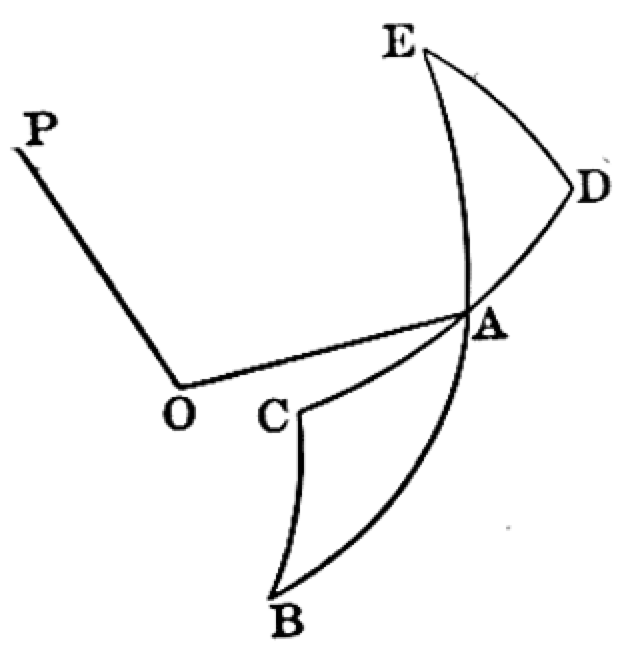
Example.—Let $\beta = li + mj + nk$, where $l^2 + m^2 + n^2 = 1$ and $R = xi + yj + zk$; then $\mathrm{S}\beta R = lx + my + nz$

$$\mathrm{V}(\beta R)\beta = \begin{vmatrix} mz - ny & nx - lz & ly - mx \\ l & m & n \\ i & j & k \end{vmatrix}$$

and
$$\mathrm{V}\beta R = \begin{vmatrix} l & m & n \\ x & y & z \\ i & j & k \end{vmatrix}.$$

Hence
$$\beta^\theta R = (lx + my + nz)(li + mj + nk) + \cos\theta \begin{vmatrix} mz - ny & nx - lz & ly - mx \\ l & m & n \\ i & j & k \end{vmatrix} + \sin\theta \begin{vmatrix} l & m & n \\ x & y & z \\ i & j & k \end{vmatrix}.$$

To prove that $\beta^b\rho$ coincides with the axis of $\beta^{-b/2}\rho^{\pi/2}\beta^{b/2}$. Take the more general versor ρ^θ. Let OP represent the axis β, AB the versor $\beta^{-b/2}$, BC the versor ρ^θ.



Then (AB)(BC) = AC = DA, therefore (AB)(BC)(AE) = (DA)(AE) = DE. Now DE has the same angle as BC, but its axis has been rotated round P by the angle b. Hence if $\theta = \pi/2$, the axis of $\beta^{-b/2}\rho^{\pi/2}\beta^{b/2}$ will coincide with $\beta^b\rho$.*

The exponential expression for

* This theorem was discovered by Cayley. It indicates that quaternion multiplication in the most general sense has its physical meaning in the composition of rotations.

$\beta^{-b/2}\rho^{\pi/2}\beta^{b/2}$ is $e^{-\frac{1}{2}b\beta^{\pi/2}+\frac{1}{2}\pi\rho^{\pi/2}+\frac{1}{2}b\beta^{\pi/2}}$, which may be expanded according to the exponential theorem, the successive powers of the trinomial being formed according to the multinomial theorem, the order of the factors being preserved.

Composition of Finite Rotations round Axes which Intersect.—Let β and γ denote the two axes in space round which the successive rotations take place, and let β^b denote the first and γ^c the second. Let $\beta^b \times \gamma^c$ denote the single rotation which is equivalent to the two given rotations applied in succession; the sign $\times$ is introduced to distinguish from the product of versors. It has been shown in the preceding paragraph that

$$\beta^b\rho = \beta^{-b/2}\rho^{\pi/2}\beta^{b/2};$$

and as the result is a line, the same principle applies to the subsequent rotation. Hence

$$\begin{aligned}\gamma^c(\beta^b\rho) &= \gamma^{-c/2}(\beta^{-b/2}\rho^{\pi/2}\beta^{\pi/2})\gamma^{c/2}\\ &= (\gamma^{-c/2}\beta^{-b/2})\rho^{\pi/2}(\beta^{b/2}\gamma^{c/2}),\end{aligned}$$

because the factors in a product of versors can be associated in any manner. Hence, reasoning backwards,

$$\beta^b \times \gamma^c = (\beta^{b/2}\gamma^{c/2})^2.$$

Let m denote the cosine of $\beta^{b/2}\gamma^{c/2}$, namely,

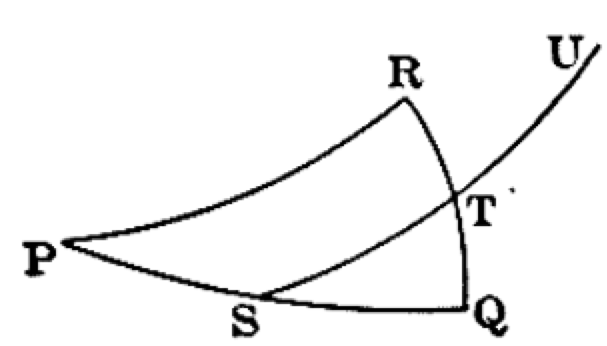
$$\cos b/2 \cos c/2 - \sin b/2 \sin c/2,$$

and $n . \nu$ their directed sine, namely,

$$\cos b/2 \sin c/2 . \gamma + \cos c/2 \sin b/2 . \beta - \sin b/2 \sin c/2 \sin \beta\gamma . \overline{\beta\gamma};$$

then $\beta^b \times \gamma^c = m^2 - n^2 + 2mn . \nu$.

Observation.—The expression $(\beta^{b/2}\gamma^{c/2})^2$ is not, as might be supposed, identical with $\beta^b\gamma^c$. The former reduces to the latter only when β and γ are the same or opposite. In the figure β^b is represented by PQ, γ^c by QR, $\beta^b\gamma^c$ by PR, $\beta^{b/2}\gamma^{c/2}$ by ST, and $(\beta^{b/2}\gamma^{c/2})^2$ by SU, which is twice ST. The cosine of SU differs from the cosine of PR by the term $-(\sin b/2 \sin c/2 \sin \beta\gamma)^2$. It is evident from the figure that their axes are also different.



Corollary.—When b and c are infinitesimals, $\cos \beta^b \times \gamma^c = 1$, and $\operatorname{Sin} \beta^b \times \gamma^c = b \,.\, \beta + c \,.\, \gamma$, which is the parallelogram rule for the composition of infinitesimal rotations.

Prob. 54. Let $\beta = \overline{30^\circ}/\underline{/45^\circ}$, $\theta = \pi/3$, and $R = 2i - 3j + 4k$; calculate $\beta^\theta R$.

Prob. 55. Let $\beta = \overline{90^\circ}/\underline{/90^\circ}$, $\theta = \pi/4$, $R = -i + 2j - 3k$; calculate $\beta^\theta R$.

Prob. 56. Prove by multiplying out that $\beta^{-b/2}\rho^{\pi/2}\beta^{b/2} = \{\beta^b \rho\}^{\pi/2}$;

Prob. 57. Prove by means of the exponential theorem that $\gamma^{-c}\beta^b\gamma^c$ has an angle b, and that its axis is $\gamma^{2c}\beta$.

Prob. 58. Prove that the cosine of $(\beta^{b/2}\gamma^{c/2})^2$ differs from the cosine of $\beta^b\gamma^c$ by $-\left(\sin\frac{b}{2}\sin\frac{c}{2}\sin\beta\gamma\right)^2$.

Prob. 59. Compare the axes of $(\beta^{b/2}\gamma^{c/2})^2$ and $\beta^b\gamma^c$.

Prob. 60. Find the value of $\beta^b \times \gamma^c$ when $\beta = \overline{0^\circ}/\underline{/90^\circ}$ and $\gamma = \overline{90^\circ}/\underline{/90^\circ}$.

Prob. 61. Find the single rotation equivalent to $i^{\pi/2} \times j^{\pi/2} \times k^{\pi/2}$.

Prob. 62. Prove that successive rotations about radii to two corners of a spherical triangle and through angles double of those of the triangle are equivalent to a single rotation about the radius to the third corner, and through an angle double of the external angle of the triangle.

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